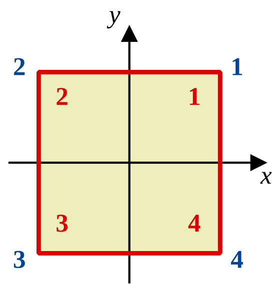
Lesson 3

3.1 The Square

To investigate the symmetries of a square, it can be placed with its centre at the origin of a Cartesian plane. The four corners of the square can be numbered to note in which quadrants they lie when in this "starting position".

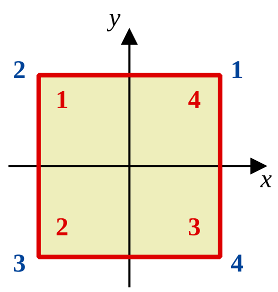


The quadrants are numbered from 1 to 4 in blue
The corners of the square are numbered from 1 to 4 in red

The first thing to do is "do nothing" !
Traditionally this is called the identity operation and is designated, *e*

3.2 Rotational Symmetry

More interesting, is to define the operation *r* as "rotate 90°"
(This will be anticlockwise, of course)
After applying *r* the situation will look like this;



Using permutation notation, $r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

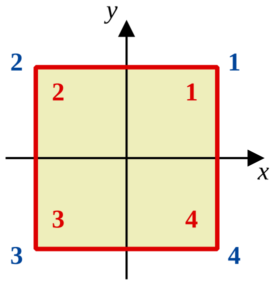
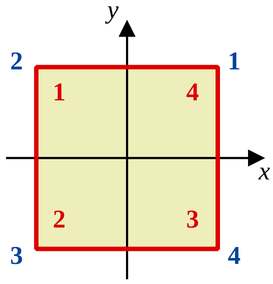
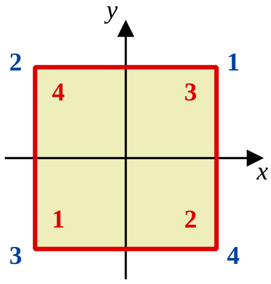
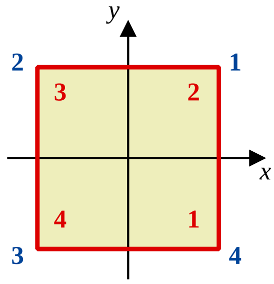
The second row shows in which quadrant each vertex is now located.

Two other symmetries of the square are "rotate 180°" and "rotate 270" which can be defined as r^2 and r^3 respectively.

Thus, for example, r^2 means "rotate 90" and then "rotate 90°" again.

Notice that $r^4 = e$

Here is a summary of the discussion so far;

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad r^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad r^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

3.3 Mirror Symmetry

In addition to rotational symmetry, a square has mirror symmetry.

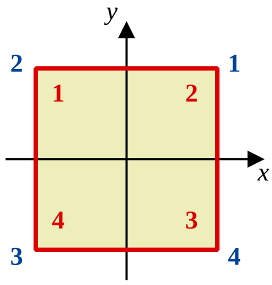
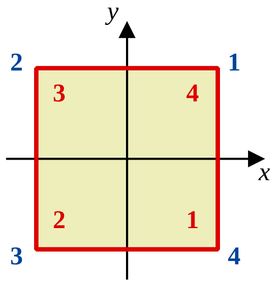
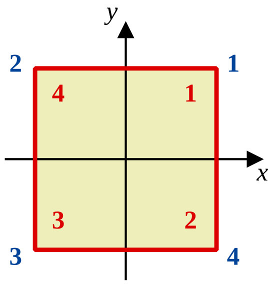
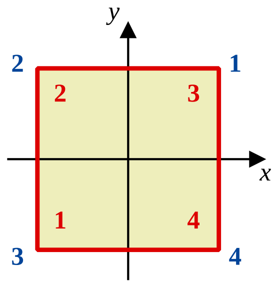
By definition let ◊ y be "reflection in the y-axis"

◊ x be "reflection in the x-axis"

◊ p be "reflection in the line with positive gradient, $y = x$"

◊ n be "reflection in the line with negative gradient, $y = -x$"

This adds four more diagrams and permutations to the proceedings;

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad n = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

Notice that, $y^2 = x^2 = p^2 = q^2 = e$

3.4 The Order of a Group

In total, eight symmetries of the square have been identified.

These will form a group of order 8, and the Cayley table for this group will be an 8 by 8 grid. The symmetries can thus be combined under the binary operation "followed by" in sixty-four different ways.

Foe example, one of the sixty-four is "rotation of 90° followed by reflection in x-axis".

The Order of a Group

The order of a group is simply the number of elements in the set upon which the binary operation acts.

3.5 An Example of Combining Symmetries

The result of combing two of the square's symmetries can be determined mathematically by using permutation notation.

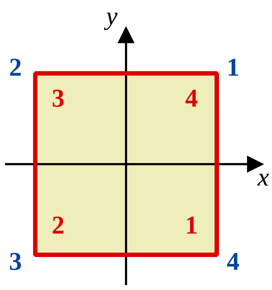
Show for a square that, using permutations, a rotation of 90° followed by a reflection in the x-axis is equivalent to a reflection in the line $y = -x$

Teaching Video: http://www.NumberWonder.co.uk/v9108/3.mp4

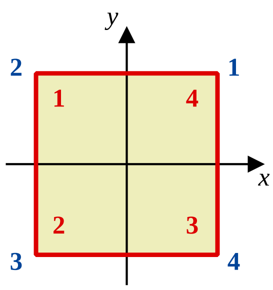
Solution :

The three symmetries involved and their associated permutations are;

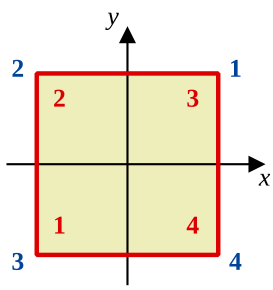
- x is a reflection in the x-axis
- r is a rotation of 90°
- n is a reflection in the line $y = -x$



$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$



$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$



$$n = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

[4 marks]

3.6 The Cayley Table for a Square

Working through all possible pairs of ways of combining the symmetries of the square gives rise to the following Cayley table.

top

then tail

$*$	e	r	r^2	r^3	y	x	p	n
e	e	r	r^2	r^3	y	x	p	n
r	r	r^2	r^3	e	n	p	y	x
r^2	r^2	r^3	e	r	x	y	n	p
r^3	r^3	e	r	r^2	p	n	x	y
y	y	p	x	n	e	r^2	r^3	r
x	x	n	y	p	r^2	e	r	r^3
p	p	x	n	y	r	r^3	e	r^2
n	n	y	p	x	r^3	r	r^2	e

This is a well known group called the dihedral group, D_4

It's alive with patterns such as,

- In any given row the eight possible symmetries occur once and once only
- In any given column the eight possible symmetries occur once and once only

Taken together this pattern is referred to as the Latin square property of a group. All groups have this Latin square property buy not all Latin squares are groups.

In reading the Cayley table notice that $x * r$ means do r first which is the r in the red row across the top of the table, then do x second which is the x in the green column down the left hand side of the table. Top then tail.

Notice that whereas $x * r = n, \; r * x = p$. That is $xr \neq rx$.

3.7 Exercise

Marks Available : 40

Question 1

Two permutations are

$$v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} \text{ and } w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$$

(i) Determine the composite permutation $v \circ w$

[2 marks]

(ii) Determine the composite permutation $w \circ v$

[2 marks]

(iii) For v determine the inverse permutation, v^{-1}

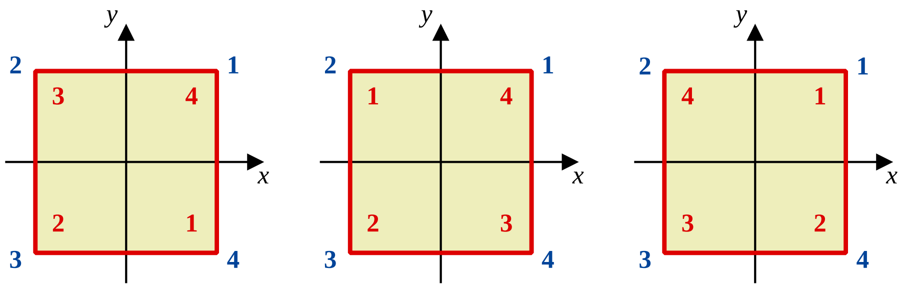
[2 marks]

Question 2

Show for a square that, using permutations, a reflection in the *x*-axis followed by a rotation of 90° is equivalent to a reflection in the line $y = x$

In answering this question label the symmetries as follows;

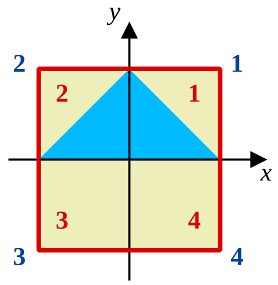
- x is a reflection in the x-axis
- r is a rotation of 90°
- p is a reflection in the line $y = x$



$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \qquad p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

[4 marks]

Question 3



The patterned square shown only has the symmetry, reflection in the *y*-axis, *y*, in addition to the "do nothing" identity, *e*.

(i) Complete the Cayley table for the group formed by $G = \{ e, y \}$ under the binary operation of composition of symmetries.

$*$	e	y
e		
y		

[1 mark]

(ii) The part (i) group is a subgroup of the group of symmetries of the (unpatterned) square.
Here it is;

$*$	e	r	r^2	r^3	y	x	p	n
e	e	r	r^2	r^3	y	x	p	n
r	r	r^2	r^3	e	n	p	y	x
r^2	r^2	r^3	e	r	x	y	n	p
r^3	r^3	e	r	r^2	p	n	x	y
y	y	p	x	n	e	r^2	r^3	r
x	x	n	y	p	r^2	e	r	r^3
p	p	x	n	y	r	r^3	e	r^2
n	n	y	p	x	r^3	r	r^2	e

What is the order of this subgroup ?

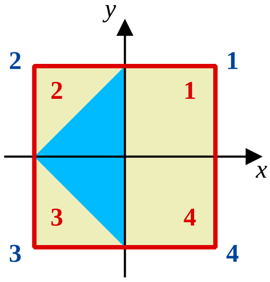
[1 mark]

Question 4

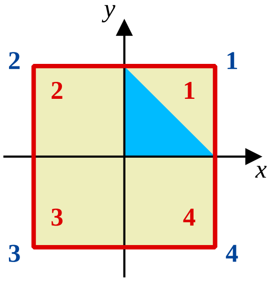
Each of the following patterned squares gives rise to a symmetry group that is a subgroup of the symmetry group of the (unpatterned) square.

For each patterned square, construct its associated Cayley table,

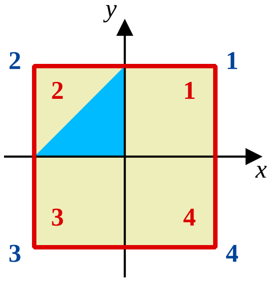
(i)



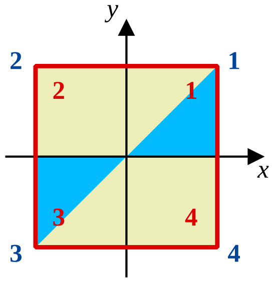
(ii)



(iii)



(iv)



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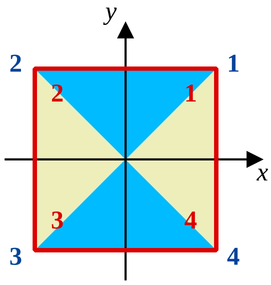
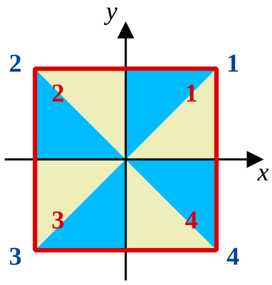
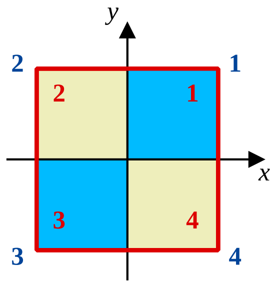


(vi)



(vii)



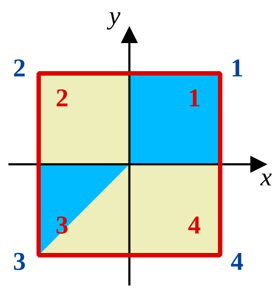
[14 marks]

Question 5

From questions 3 and 4, you should have unearthed that the symmetry group of the square, (which is of order 8), has five subgroups of order 2, and three subgroups of order 4.

The identity element on its own can also be considered a subgroup corresponding to a patterned square with no symmetry, such as the example shown below.

It adds one subgroup of order 1 to the list of subgroups of the (unpatterned) square.



Finally, the whole group is considered to be a subgroup of itself.

That concession adds one subgroup or order 8.

Here is a summary of all subgroups found;

Subgroup	Order	Number Found
$\{e\}$	1	1
$\{e, x\}$, $\{e, y\}$, $\{e, p\}$, $\{e, n\}$, $\{e, r^2\}$	2	5
$\{e, r, r^2, r^3\}$ $\{e, r^2, x, y\}$ $\{e, r^2, p, n\}$	4	3
$\{e, r, r^2, r^3, x, y, p, n\}$	8	1

There is a £5,000,000 prize for the person who finds a subgroup of the symmetries of a square of order 3, 5, 6 or 7. It can never be claimed !

What rule does the table of results suggest holds between the order of a group and the order of its subgroups ?

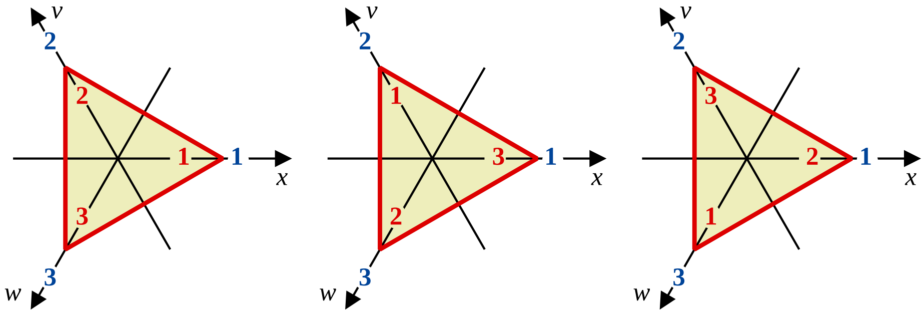
This rule is called Lagrange's Theorem.

[2 marks]

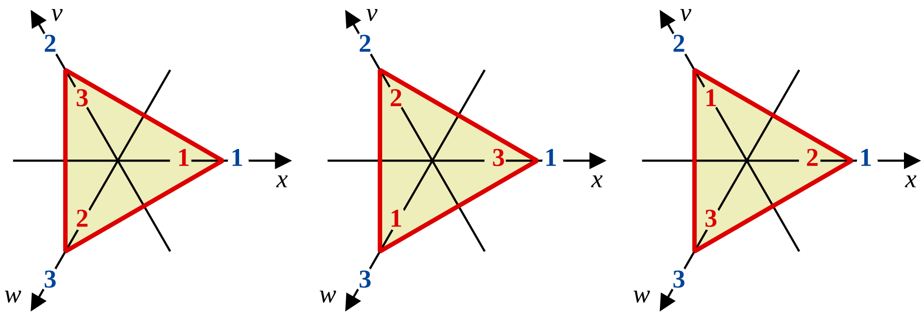
Question 6

The diagram shows the six symmetries of an equilateral triangle where

- ◊ e is “do nothing”
- ◊ r is rotation of 120°
- ◊ r^2 is rotation of 240°
- ◊ x is “reflection in the x-axis”
- ◊ v is “reflection in the v-axis”
- ◊ w is “reflection in the w-axis”



$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $r = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $r^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$



$x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $v = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Construct a Cayley table for the group of symmetries of the equilateral triangle

*	e	r	r^2	x	v	w
e						
r						
r^2						
x						
v						
w						

[6 marks]

Question 7

For the square, the following table of all possible subgroups was constructed,

Subgroup	Order	Number Found
$\{e\}$	1	1
$\{e, x\}, \{e, y\},$ $\{e, p\}, \{e, n\},$ $\{e, r^2\}$	2	5
$\{e, r, r^2, r^3\}$ $\{e, r^2, x, y\}$ $\{e, r^2, p, n\}$	4	3
$\{e, r, r^2, r^3, x, y, p, n\}$	8	1

Construct a similar table for all possible subgroups of the equilateral triangle.

[6 marks]