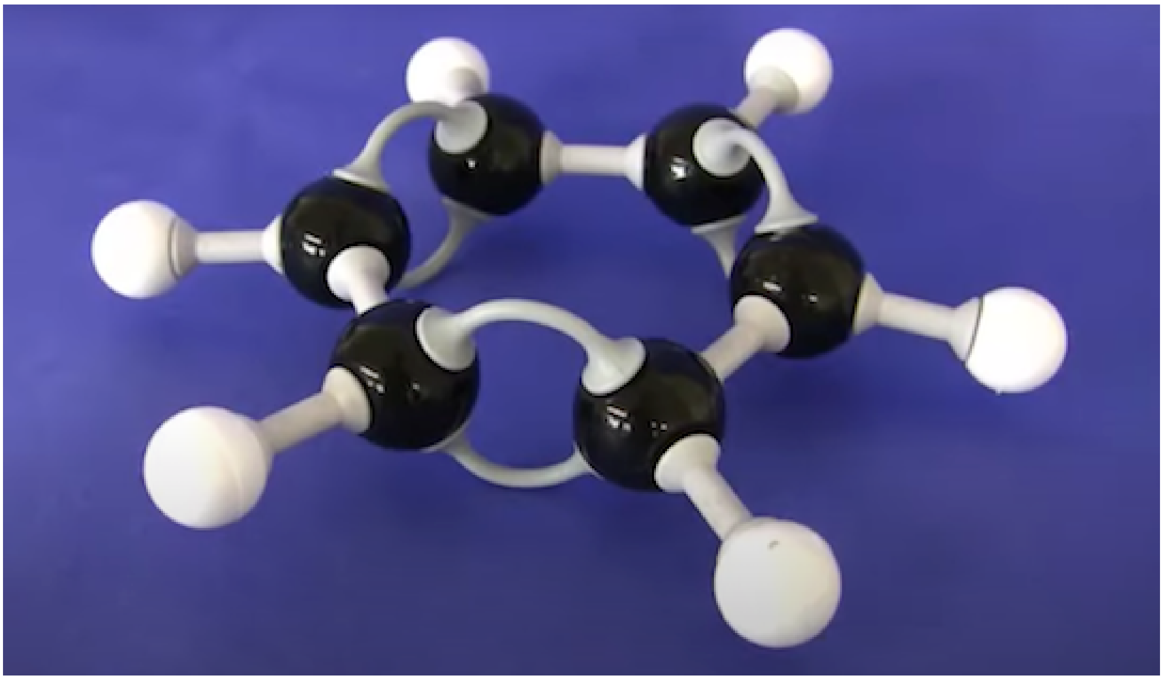
Lesson 5

5.1 Isomorphism

One aim of group theory is to capture the symmetries of various entities, be they, for example, a geometric shape, a set of functions, or a set of numbers under modulo arithmetic. However, it's reach is broader than that, Recently, for example, group theory predicted the existence of a tiny particle in physics. Knowing roughly what to look for, this was then detected in the physical world. Another application of group theory is in the study of crystals in Chemistry.



A model of the molecule Benzine

Group theory reveals that there are limits on the types of structure that symmetrical objects can have. For example, when the order of a group is a prime number, there is only one fundamental group of that order and that it has to be cyclic. Seemingly, different groups of that order have the same underlying structure; mathematicians say they are "isomorphic". As a further example, there are only two possible groups of order 6, one cyclic and one not. So any group of order 6 has to be isomorphic to one of those two groups. Thus, what may have initially seems like a bewildering array of groups is actually more constrained. Group theory is the tool that can identify what those constraints are.

5.2 A Catalogue of Possible Groups

On the next page is a catalogue of all possible groups of order less than 12. Faced with an unknown group of order less than 12, the natural question to ask is which group from this catalogue is the unknown group isomorphic to. Spotting an isomorphism using Cayley tables can be tricky; the catalogue lists each group's defining properties. Matching these is often the easier option.

Order	Name	Examples	Properties
1	$\mathbb{Z}_1$	Trivial group	Only group of order 1
2	$\mathbb{Z}_2$	$\{0, 1\}$ under $+_2$	Only group of order 2
3	$\mathbb{Z}_3$	$\{0, 1, 2\}$ under $+_3$	Only group of order 3
4	$\mathbb{Z}_4$	$\{0, 1, 2, 3\}$ under $+_4$	Cyclic group of order 4
	K_4	Klein group Symmetry group of a rectangle	Not cyclic. Every element (except identity) has order 2
5	$\mathbb{Z}_5$	$\{0, 1, 2, 3, 4\}$ under $+_5$	Cyclic group of order 5
6	$\mathbb{Z}_6$	$\{0, 1, 2, 3, 4, 5\}$ under $+_6$	Cyclic group of order 6
	S_3, D_4	All permutations of 3 objects, Equilateral triangle's symmetry	Not cyclic $\therefore$ no element of order 6
7	$\mathbb{Z}_7$	$\{0, 1, 2, 3, 4, 5, 6\}$ under $+_7$	Cyclic group of order 7
8	$\mathbb{Z}_8$	$\{0, 1, 2, 3, 4, 5, 6, 7\}$ under $+_8$	Cyclic group of order 8
	D_4	Symmetry group of a square (Some text books call this D_8)	Not cyclic $\therefore$ no element of order 8 Exactly 2 elements of order 4
	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Rational points on the elliptic curve $y^2 = (x+3)x(x-1)$	Not cyclic $\therefore$ no element of order 8 Exactly 4 elements of order 4
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Symmetry group of a cuboid	Not cyclic $\therefore$ no element of order 8 Every element (except identity) has order 2
	Q_4	Quaternion group	Not cyclic $\therefore$ no element of order 8 Exactly 6 elements of order 4
9	$\mathbb{Z}_9$	$\{0, 1, 2, 3, 4, ..., 8\}$ under $+_9$	Cyclic group of order 9
	$\mathbb{Z}_3 \times \mathbb{Z}_3$		Not cyclic $\therefore$ no element of order 9 Exactly 4 elements of order 3
10	$\mathbb{Z}_{10} = \mathbb{Z}_5 \times \mathbb{Z}_2$	$\{0, 1, 2, 3, 4, ..., 9\}$ under $+_{10}$	Cyclic group of order 10
	D_5	Frobenius group (Some text books call this D_{10})	Not cyclic $\therefore$ no element of order 10
11	$\mathbb{Z}_{11}$	$\{0, 1, 2, 3, 4, ..., 10\}$ under $+_{11}$	Cyclic group of order 11

5.3 Example

The Cayley tables for two groups of order 6 are presented below.
The first is for multiplication modulo 7 on the set {1, 2, 3, 4, 5, 6}
The second is for addition modulo 6 on the set {0, 1, 2, 3, 4, 5}
Are these two groups isomorphic or not ?

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Teaching Video: http://www.NumberWonder.co.uk/v9108/5.mp4

The video talks through to solution presented on the next page

[6 marks]

5.4 Where Do Mathematicians' Go When They Die ?

The Symmetry

5.5 Solution to the 5.3 Example

Start by identifying each group's identity element which is easy from a Cayley table; simply look for the row that matches the column headings.
Circle all locations of the identity in both tables.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

At this stage it's not clear if the two are isomorphic or not.
If an isomorphism, *f*, exists it must map one identity onto the other.

$$f(1) = 0$$

From the catalogue of groups it is known that there are two groups of order 6.
One is cyclic and one is not.
The catalogue also states that the set {0, 1, 2, 3, 4, 5} under $+_6$ is cyclic.
(That is, under addition, modulo 6)
So if it can be shown that the multiplication modulo 7 group is also cyclic then the two must be isomorphic. On the other hand, if the multiplication modulo 7 group is not cyclic, then no isomorphism between the two exists.

If the modulo 7 group is cyclic it will be possible to generate the entire group from at least one of it's elements.
Starting to work through {1, 2, 3, 4, 5, 6 } using each element as a generator;

$\langle 1 \rangle = \{1\}$

$\langle 2 \rangle = \{1, 2, 4\}$

$\langle 3 \rangle = \{1, 2, 3, 4, 5, 6\}$ $\therefore$ The modulo 7 group is cyclic

$\therefore$ The two groups are isomorphic.

The question does not ask for any more details but, out of interest, here is the isomorphism made plain by manipulating one of the Cayley tables;

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$+_6$	0	2	1	4	5	3
0	0	2	1	4	5	3
2	2	4	3	0	1	5
1	1	3	2	5	0	4
4	4	0	5	2	3	1
5	5	1	0	3	4	2
3	3	5	4	1	2	0

5.6 Describing an Isomorphism Mathematically

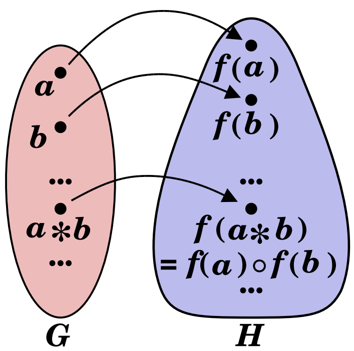
Intuitively, isomorphic groups have the same symmetry, structured in the same way. Given that an isomorphism exists between two groups G and H, all of the key properties of G are to be found in H such as, for example, the number and order of the various subgroups and the number and order of the various elements. In particular, if G is cyclic then so too is H.

Formally, an isomorphism is described as follows;

Definition : Isomorphism

Two groups $(G, *)$ and $(H, \circ)$ are isomorphic, written $G \cong H$, if there exists a mapping f that maps G onto H in the following manner;

- f maps all of the elements of G onto all of the elements of H
- f maps each element of G to a different element of H
- f preserves the structure $f(a * b) = f(a) \circ f(b)$

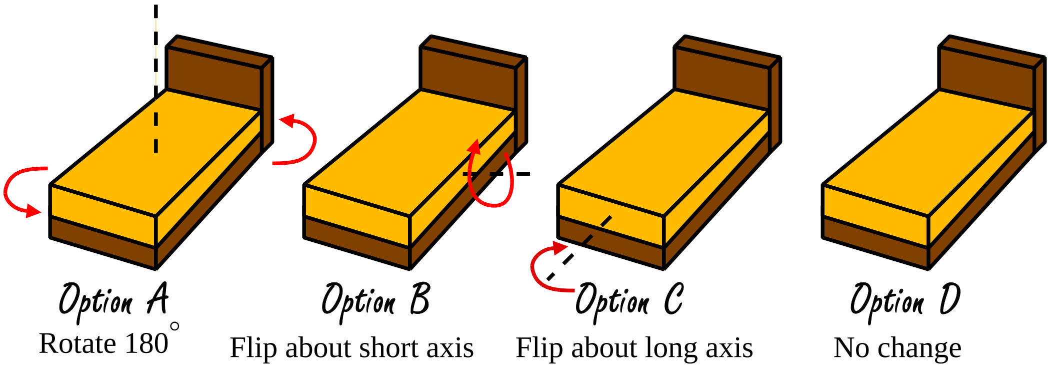


5.7 Exercise

Marks Available: 40

Question 1

A mattress manufacturer suggests that customers "flip" their mattress regularly so that it wears out evenly. The following instructions are provided:



The operation $\circ$ is defined on $\{A, B, C, D\}$ as "followed by" so that, for example, $A \circ C$ means "flip about the long axis followed by rotate 180°"

(i) Complete the Cayley table for these four options;

$\circ$	***A***	***B***	***C***	***D***
A				
B		***D***		
C	***B***			
D				***D***

[3 marks]

(ii) Which element is the identity element ?

[1 mark]

There are two groups of order 4.
One is the cyclic group $\mathbb{Z}_4$, the other is the (non-cyclic) Klein group, K_4

(iii) Work out the order of each of the elements A, B, C and D.

[2 marks]

(iv) Is the group of mattress transformations cyclic ?

[1 mark]

(v) State if the mattress transformations group is isomorphic to $\mathbb{Z}_4$ or K_4

[1 mark]

Question 2

The Catalogue of Possible Groups list five fundamental groups of order 8.

They are $\mathbb{Z}_8$, D_4, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and Q_4

Shown below is the Cayley table for an unknown group of order 8.

$*$	s	a	k	m	w	z	c	p
s	m	z	w	s	k	a	p	c
a	z	m	p	a	c	s	w	k
k	w	p	m	k	s	c	z	a
m	s	a	k	m	w	z	c	p
w	k	c	s	w	m	p	a	z
z	a	s	c	z	p	m	k	w
c	p	w	z	c	a	k	m	s
p	c	k	a	p	z	w	s	m

(i) Which element is the identity element ?

[1 mark]

(ii) List the inverse of each element.

[2 marks]

(iii) Is the Cayley table of a cyclic group ?
Give a reason for your answer.

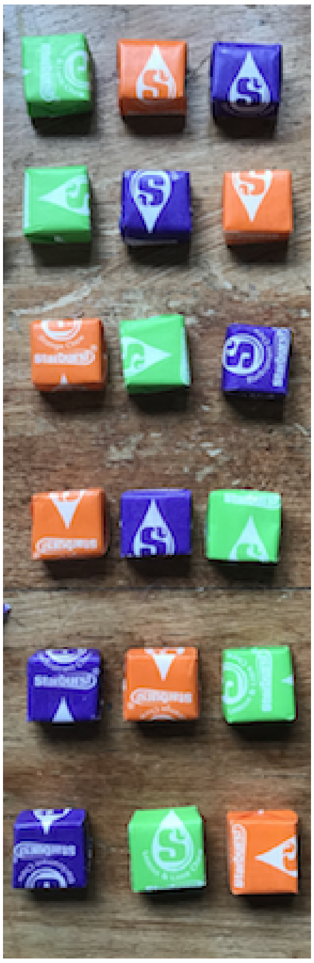
[2 marks]

(iv) To which of the possible fundamental groups of order 8 is the unknown group isomorphic ?
Give a reason for your answer.

[2 marks]

Question 3

Michael has a bag of Starburst™ sweets and has worked out that there are six different ways of arranging one each of the colours Green, Orange, and Purple.



$$e = \begin{pmatrix} G & O & P \\ G & O & P \end{pmatrix}$$

$$r = \begin{pmatrix} G & O & P \\ G & P & O \end{pmatrix}$$

$$s = \begin{pmatrix} G & O & P \\ O & G & P \end{pmatrix}$$

$$t = \begin{pmatrix} G & O & P \\ O & P & G \end{pmatrix}$$

$$u = \begin{pmatrix} G & O & P \\ P & O & G \end{pmatrix}$$

$$v = \begin{pmatrix} G & O & P \\ P & G & O \end{pmatrix}$$

Michael suspects that the set $\{e, r, s, t, u, v\}$ under the binary operation "followed by" forms a group $(G, *)$ where, for example, $s * t$ means t followed by s.

(i) Show that $s * t = r$ by using two-row notation

[2 marks]

(ii) Show that $s * t \neq t * s$

[2 marks]

(iii) What is the inverse of the element s ?
Is it e, r, s, t, u or v ?
Give a reason for your answer.

[2 marks }

(iv) Construct the Cayley table for the proposed group $(G, *)$

$*$	e	r	s	t	u	v
e	e	r	s			
r	r	e				
s	s					
t						e
u					e	r
v				e	s	t

[4 marks]

(v) Show that $(G, *)$ is a group. You may assume associativity.

[3 marks]

(vi) Which of the groups of order 6, $\mathbb{Z}_6$ or S_3, is G isomorphic to ?

Give a reason for your answer.

[2 marks]

Question 4

The set $G = \{1, 7, 11, 13, 17, 19, 23, 29\}$ forms a group under multiplication modulo 30.

(i) Find the order of each element of this group.

[4 marks]

(ii) Find a cyclic subgroup of G of order 4

[2 marks]

(iii) Find a subgroup of G which is isomorphic to the Klein four-group.

[2 marks]

The set $H = \{0, 1, 2, 3, 4, 5, 6, 7\}$ forms a group under addition modulo 8.

(iv) State, with reasons, whether $G \cong H$

[2 marks]