2.1 A Formal Definition

The first lecture informally introduced vector spaces and subspaces. In particular the fact that a basis underpinned the structure was established, along with an emphasis on the additive nature of scalar multiples of the basis vectors. This was presented as the requirement that linear combinations of the basis vectors formed a system with the property of closure and that “the normal rules of arithmetic” applied. However, a formal definition is now needed to tighten up on exactly what a vector space is.

Definition: Vector Space

A vector space consists of a set V of elements over a field $\mathbb{F}$ such that,

- $(V, +)$ is an Abelian group
- given any vectors $\boldsymbol{v}. \boldsymbol{w} \in V$ and $a, b \in \mathbb{F}$
    - $a\boldsymbol{v} \in V$ (multiplicative closure)
    - $1\boldsymbol{v} = \boldsymbol{v}$ (multiplicative identity)
    - $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$ (multiplicative associativity)
    - $a(\boldsymbol{v} + \boldsymbol{w}) = a\boldsymbol{v} + a\boldsymbol{w}$ (multiplicative left distributivity)
    - $(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$ (multiplicative right distributivity)

Definition: Group

If V is a set and $+$ is a binary operation defined on V, then $(V, +)$ is a group if the following four axioms hold:

- **Closure:** For all $\boldsymbol{v}, \boldsymbol{w} \in V$, $\boldsymbol{v} + \boldsymbol{w} \in V$
- **Identity:** There exists an identity element, a “zero vector”, $\boldsymbol{0} \in V$

  This is such that, for all $\boldsymbol{v} \in V$, $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v}$
- **Inverses:** For each $\boldsymbol{v} \in V$, there exists an inverse element $-\boldsymbol{v} \in V$

  This is such that $\boldsymbol{v} + -\boldsymbol{v} = -\boldsymbol{v} + \boldsymbol{v} = 0$
- **Associativity:** For all $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z} \in V$, $\boldsymbol{v} + (\boldsymbol{w} + \boldsymbol{z}) = (\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{z}$

Definition: Abelian Group

If V is a set and $+$ is a binary operation defined on V, and $(V, +)$ is a group, then $(V, +)$ is an Abelian group if,

- **Commutativity:** For all $\boldsymbol{v}, \boldsymbol{w} \in V$, $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$

2.2 Examples of Vector Spaces

Working through all of the conditions to determine if a proposed vector space satisfies them all is tedious. Fortunately the following can be quoted as "known":

(i) $\mathbb{R}^n$: The set of all n-dimensional column vectors with real elements forms a real vector space under vector addition and scalar multiplication.

(ii) $\mathbb{C}^n$: The set of all n-dimensional column vectors with complex elements forms a complex vector space under vector addition and scalar multiplication.

(iii) $\mathbb{P}_n$: The set of all polynomials whose degree is less than n with polynomial addition and scalar multiplication. This vector space is n-dimensional.

(iv) $\mathbb{Z}_p^n$: The set of all n-dimensional column vectors with integer elements forms an integer vector space under addition and scalar multiplication modulo p where p is a prime number.

(v) $\mathbb{R}^{m \times n}$: The set of all $m \times n$ matrices with real entries forms a real mn-dimensional vector space with matrix addition and scalar multiplication.

(vi) $\mathbb{C}^{m \times n}$: The set of all $m \times n$ matrices with complex entries forms a complex mn-dimensional vector space with matrix addition and scalar multiplication.

(vii) The set of all infinite sequences forms a complex vector space under corresponding term-by-term addition and scalar multiplication.

2.3 Vector Subspaces

Given the above list, exam questions tend to focus on deciding if a non-empty subset of one of the above is itself a vector space.

Testing for a Vector Subspace

Let V be a vector space over a field $\mathbb{F}$ and let S be a non-empty subset of V.

S is a vector subspace of V if and only if S satisfies the following conditions,

(1) Additive Identity : $0 \in S$

(2) Closed under Addition : $s, t \in S$ implies $s + t \in S$

(3) Closed under scalar multiplication : $a \in \mathbb{F}$ and $s \in S$ implies $as \in S$

2.4 Example

Consider a proposed vector space of 2×2 symmetric matrices $\mathbf{M}$ with real entries.

That is, matrices satisfying $\mathbf{M}_{ij} = \mathbf{M}_{ji}$

(i) Show that the proposed vector space is indeed a vector space.

[3 marks]

(ii) Write down an explicit basis for this vector space.

[2 marks]

(iii) What is the dimension of this vector space ?

[1 mark]

Teaching Video : http://www.NumberWonder.co.uk/v9112/2.mp4

<= The video will talk through the solution to the example

(i) It is known that the set of all $m \times n$ matrices with real entries forms a real mn-dimensional vector space with matrix addition and scalar multiplication. Therefore, it only needs to be shown that the vector space of 2×2 symmetric matrices with real entries is a vector subspace of this larger vector space.

(1) Observe that, $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which is symmetric.

As required, the zero element of the parent is in the vector subspace.

(2) Let two general 2×2 symmetric matrices be,

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$$

in which case, $\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_2 + b_2 & a_3 + b_3 \end{pmatrix}$ which is symmetric.

As required, there is closure under addition.

(3) Let a general 2×2 symmetric matrix and a general scalar be,

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \text{ and } s$$

in which case, $s\mathbf{A} = s\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} = \begin{pmatrix} sa_1 & sa_2 \\ sa_2 & sa_3 \end{pmatrix}$ which is symmetric.

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace of 2×2 symmetric matrices with real entries is indeed a vector subspace of the set of all $m \times n$ matrices with real entries.

It can be considered to be a vector space in its own right. □

[3 marks]

(ii) A suitable basis for the vector space of 2×2 symmetric matrices with real entries could be,

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

[2 marks]

(iii) The vector space of 2×2 symmetric matrices with real entries is a 3-dimensional vector space, because the basis contains three vectors.

[1 mark]

2.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

The set of all infinite sequences forms a real vector space over the field of real numbers under corresponding term-by-term addition and scalar multiplication. Consider a proposed vector subspace of geometric progressions with common ratio of 3. Show that the proposed vector subspace is indeed a vector subspace.

[3 marks]

Question 2

The set of points $(x, y, z) \in \mathbb{R}^3$ forms a real vector space over the field of reals.
Consider a proposed vector subspace defined by the restriction $x + y + z = 0$
Show that this proposed vector subspace is indeed a vector subspace.

[3 marks]

Question 3

The set of points $(x, y, z) \in \mathbb{R}^3$ forms a real vector space over the field of reals.

Consider a proposed vector subspace defined by the restriction $x + y + z = 1$

Show that the proposed vector subspace is not a vector subspace.

[3 marks]

Question 4

Show that all polynomials whose degree is equal to three do not form a vector space over the field of reals with polynomial addition and scalar multiplication.

[3 marks]

Question 5

Given a matrix **M**, the transpose of **M**, written $\mathbf{M}^{\mathrm{T}}$, is an operator which flips the matrix over on its diagonal: rows become columns and columns become rows.

For example, $\begin{pmatrix} 1 & 5 & -6 \\ -2 & -4 & 3 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & -2 \\ 5 & -4 \\ -6 & 3 \end{pmatrix}$

Properties of the Transpose

- A square matrix, **S**, is considered orthogonal iff $\mathbf{S}^{\mathrm{T}} \times \mathbf{S} = \mathbf{S} \times \mathbf{S}^{\mathrm{T}} = \mathbf{I}$
- A square matrix, **S**, is symmetric iff $\mathbf{S}^{\mathrm{T}} = \mathbf{S}$
- A square matrix, **S**, is antisymmetric iff $\mathbf{S}^{\mathrm{T}} = -\mathbf{S}$
- The symmetric part of any square matrix **M** is $\frac{1}{2}\left(\mathbf{M} + \mathbf{M}^{\mathrm{T}}\right)$
- The antisymmetric part of any square matrix **M** is $\frac{1}{2}\left(\mathbf{M} - \mathbf{M}^{\mathrm{T}}\right)$
- $(\mathbf{A} \pm \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} \pm \mathbf{B}^{\mathrm{T}}$
- $\left(\mathbf{M}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{M}^{-1}\right)^{\mathrm{T}}$ provided $det(\mathbf{M}) \neq 0$

Explain why,

(i) Any square matrix is the sum of its symmetric and antisymmetric parts.

[2 marks]

(ii) Any antisymmetric matrix must have zeros on its diagonal.

[2 marks]

(iii) Any 3×3 antisymmetric matrix is singular.

[3 marks]

Question 6

Consider the vector space of 3×3 matrices $\mathbf{W}$ with real entries $\mathbf{W}_{ij}$

(i) What is the dimension of this vector space ?
Give a reason for your answer.

[2 marks]

A real antisymmetric matrix $\mathbf{W}$ has real entries such that $\mathbf{W}_{ij} = -\mathrm{W}_{ji}$

(ii) Show that the real antisymmetric 3×3 matrices form a vector subspace.

[3 marks]

(iii) Write down an explicit basis for this vector subspace.

[2 marks]

(iv) What is the dimension of this vector subspace ?

[1 mark]

(v) Given that the real antisymmetric $n \times n$ matrices form a vector subspace for any given positive integer value of n, what is the dimension of the vector subspace. Give your answer as a formula in terms of n.

[2 marks]

Question 7

(i) Use matrix methods and a calculator (or other device) that can handle matrices to show that any vector in $\mathbb{R}^3$ can be written as a linear combination of the following basis vectors,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

[5 marks]

(ii) Write a general vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ as a linear combination of this basis in terms of x, y and z.

[2 marks]

Question 8

Let V be the vector space of all real 2×2 square matrices.

Let $\mathbf{A}$ be an orthogonal real 2×2 matrix; one where $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$

Consider the subset, $W := \{ \mathbf{A} \in V \mid \mathbf{A} \text{ is an orthogonal matrix} \}$

Is the set of all orthogonal matrices, W, a vector subspace of V ?

Justify your answer.

[5 marks]

Question 9

Show that the proposed basis, $B = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is not suitable for the vector space $\mathbb{Z}_p^2$, where p is a prime, but is suitable for the vector space $\mathbb{R}^2$.

[5 marks]

Question 10

A proposed vector subspace of $\mathbb{R}^4$ is based upon the following set,

$$S = \left\{ \begin{pmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{pmatrix} \middle| \; a,\, b,\, c \in \mathbb{R} \right\}$$

Find a suitable basis for S and state the dimension of the resulting vector subspace.

[4 marks]