3.1 Merging Vector Subspaces

Having established what a vector subspace is, and how to determine whether a proposed vector subspace is or is not a vector subspace, there are various ways of merging two vector subspaces. In this work it will be assumed that the two vector subspaces to be merged are both subspaces of the same vector space.

Suppose that a vector space V has two subspaces U and W.
One type of merge of particular interest is the sum of the subspaces, $U + W$.
This is itself a vector subspace, as will be proven in the exercises.
It is the set of all possible sums $\boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$.

If the basis of U and W are,

$$B_U = \{ \boldsymbol{u}_{b1}, \boldsymbol{u}_{b2}, \ldots, \boldsymbol{u}_{bn} \} \text{ and } B_W = \{ \boldsymbol{w}_{b1}, \boldsymbol{w}_{b2}, \ldots, \boldsymbol{w}_{bm} \}$$

then the basis of $U + W$ is buried somewhere in the list of vectors,

$$\boldsymbol{u}_{b1}, \boldsymbol{u}_{b2}, \ldots, \boldsymbol{u}_{bn}, \boldsymbol{w}_{b1}, \boldsymbol{w}_{b2}, \ldots, \boldsymbol{w}_{bm}$$

Definition : Span

Given a list of vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n,$ the set of all linear combinations of those vectors is termed their span, and is denoted $span(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n)$

The problem the list of vectors presents is that it may not be linearly independent and so may not be a ready made basis for the vector subspace $U + W$.
It may be that it's easy to spot that one of the vectors in the list is a multiple of another, in which case it can be removed, but with a list of more than two vectors it's possible for none to be a multiple of one other and yet still the system is not linearly independent because one is a linear combination of the others.

Definition: Linearly independent

Given a list of vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n$ in V over a field $\mathbb{F}$, if the only choice of $a_1, a_2, a_3, \ldots, a_n \in \mathbb{F}$ that makes $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 + \ldots + a_n\boldsymbol{v}_n$ equal to zero is $a_1 = a_2 = a_3 = \ldots = a_n = 0$ then the list of vectors are said to be linearly independent.

3.2 Example

(i) Show that the following list of vectors is not linearly independent over $\mathbb{R}^3$,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

(ii) Remove a vector and show the remaining two are linearly independent.

[4, 2 marks]

Teaching Video: http://www.NumberWonder.co.uk/v9112/3.mp4

<= The video will present a solution to the example

[6 marks]

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Let V be a real vector space over the field $\mathbb{R}^3$
Let U and W be the vector subspaces with basis,

$$B_U = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \ B_W = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(i) Describe geometrically the following vector subspaces of V,

(a) U

[1 mark]

(b) W

[1 mark]

(c) $U \cap W$

[1 mark]

(ii) **(a)** Write down a basis for $U + W$
(Be sure to check that it is linearly independent)

[2 marks]

(b) Describe geometrically the vector subspace $U + W$

[2 marks]

Question 2

Let V be a real vector space over the field $\mathbb{R}^3$

Let U and W be the vector subspaces with basis,

$$B_U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \; B_W = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(i) Describe geometrically the following vector subspaces of V,

(a) U

[2 mark]

(b) W

[2 mark]

(c) $U \cap W$

[2 mark]

(ii) **(a)** Write down a basis for $U + W$

(Be sure to check that it is linearly independent)

[2 marks]

(b) Describe geometrically the vector subspace $U + W$

[2 marks]

Question 3

Let V be a real vector space over the field $\mathbb{R}^3$

Let U and W be the vector subspaces with basis,

$$B_U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \ B_W = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(i) Describe geometrically the following vector subspaces of V,

(a) U

[3 mark]

(b) W

[3 mark]

(c) $U \cap W$

[3 mark]

(ii) **(a)** Write down a basis for $U + W$
(Be sure to check that it is linearly independent)

[4 marks]

(b) Describe geometrically the vector subspace $U + W$

[2 marks]

Question 4

Consider a real vector space V and two vector subspaces U and W of V

(a) Prove that the intersection $U \cap W$ is a vector subspace of V

[6 marks]

(b) Prove that $U + W$ is a vector subspace of V

($U + W$ is the set of all sums $\boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$)

[6 marks]

(c) The dimensions of the above vector spaces are related by,

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$

Verify this formula for the specific example where $V = \mathbb{R}^3$, and where

- U is spanned by $\boldsymbol{u}_1 = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{u}_2 = \boldsymbol{k}$
- W is spanned by $\boldsymbol{w}_1 = \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{w}_2 = -\boldsymbol{i} + 2\boldsymbol{j}$

[6 marks]