**Lesson 2**

**A-Level Pure Mathematics : Year 2**
**Integration III**

## 2.1 Integration by Parts

This is a product rule for integration.
Use it when a product is to be integrated, for example;

$$\int x \ \sin x \ dx$$

Integration by parts expands such integrals into four pieces. It requires some differentiation, ***D***, as well as integration, ***I***, and some leaving alone, ***L***.

The mnemonic ***L I D I*** may help. (*Lie -Die*)

In the examples we'll be integrating and differentiating $\sin x$, $\cos x$, $\ln x$ and $e^x$
Here is a reminder of their derivatives:

| $f(x)$ | $f'(x)$ |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\ln x$ | $\frac{1}{x}$ |
| $e^x$ | $e^x$ |

**Example N° 1**

$$\int x \ \sin x \ dx$$

$$= \ \boldsymbol{L}(x) \ \ \boldsymbol{I}(\sin x) - \int \ \boldsymbol{D}(x) \ \ \boldsymbol{I}(\sin x) \ dx$$

$$= \ \ x \ \ \ (-\cos x) - \int \ 1 \ \ (-\cos x) \ \ dx$$

$$= -x \ \cos x + \int \cos x \ dx$$

$$= -x \ \cos x + \sin x + c$$

**[ 3 marks ]**

**Example N° 2**

$$\int x^3 \ln x \ dx$$

First, swap the order

as we can $\boldsymbol{D}(\ln x)$ but not (yet!) $\boldsymbol{I}(\ln x)$

$$= \int \ln x \ x^3 \ dx$$

$$= \boldsymbol{L}(\ln x) \ \boldsymbol{I}\left(x^3\right) - \int \boldsymbol{D}(\ln x) \ \boldsymbol{I}\left(x^3\right) \ dx$$

$$= \ln x \quad \frac{x^4}{4} \quad - \int \quad \frac{1}{x} \quad \frac{x^4}{4} \quad dx$$

$$= \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} \ dx$$

$$= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + c$$

**[ 3 marks ]**

**Example N° 3**

$$\int \ln x \ dx$$

Sneaky question because this

does not look like a product

$$= \int \ln x \ \times 1 \ dx$$

$$= \boldsymbol{L}(\ln x) \ \boldsymbol{I}(1) - \int \boldsymbol{D}(\ln x) \ \boldsymbol{I}(1) \ dx$$

$$= \ln x \quad x \quad - \int \quad \frac{1}{x} \quad x \quad dx$$

$$= x \ln x - \int 1 \ dx$$

$$= x \ln x - x + c$$

A mystery is solved; $\ln x$ can be integrated as well as differentiated !

**[ 3 marks ]**

## 2.2 Exercise

*Any solution based entirely on graphical or numerical methods is not acceptable*

Marks Available : 50

**Question 1**

Find the integral;

$$\int x \cos x \, dx$$

**[ 3 marks ]**

**Question 2**

Find the integral;

$$\int x \sin 4x \, dx$$

**[ 4 marks ]**

**Question 3**

This is the only question in the exercise were it's necessary to swap the order of the product in order to determine the integral, like Example N° 2

$$\int x^5 \ln x \, dx$$

**[ 3 marks ]**

**Question 4**
Find the integral;

$$\int x\ e^{x}\ dx$$

**[ 3 marks ]**

**Question 5**
Find the integral;

$$\int x\ e^{-5x}\ dx$$

**[ 4 marks ]**

**Question 6**
Find the integral;

$$\int \frac{x}{2\ e^{x}}\ dx$$

**[ 4 marks ]**

**Question 7**

**( i )** By setting up a chain rule backwards, find

$$\int (3x + 1)^6 \, dx$$

**[ 2 marks ]**

**( ii )** Use your part (i) answer and integration by parts to show that

$$\int x (3x + 1)^6 \, dx = \frac{x (3x + 1)^7}{21} - \frac{(3x + 1)^8}{21 \times 24} + c$$

**[ 4 marks ]**

**( iii )** Simplify your answer by showing that

$$\frac{x (3x + 1)^7}{21} - \frac{(3x + 1)^8}{21 \times 24} + c = \frac{1}{504} (3x + 1)^7 (21x - 1) + c$$

**[ 3 marks ]**

**Question 8**

Evaluate, giving an exact answer,

$$\int_0^{\frac{\pi}{3}} x \ sin\, 3x \ dx$$

**[ 6 marks ]**

**Question 9**

Evaluate giving an exact answer,

$$\int_0^1 (2x + 1)\, e^x \ dx$$

**[ 6 marks ]**

**Question 10**

Do not use integration by parts at any stage in this question !

Instead, use the substitution $u = 3x - 5$ to evaluate,

$$7 \int_1^2 x^2 \, (3x - 5)^4 \, dx$$

**[ 8 marks ]**