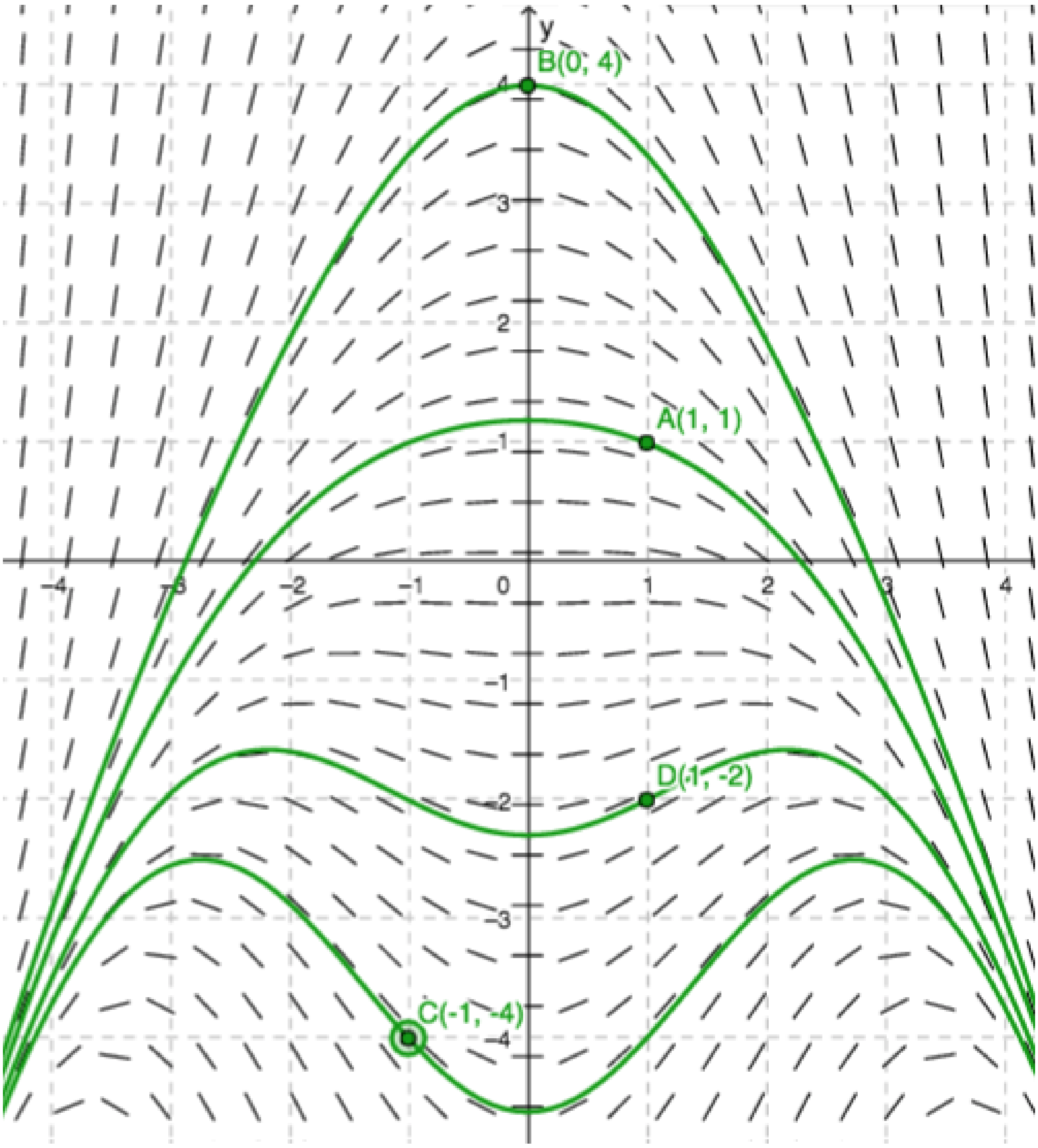
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

DIFFERENTIAL EQUATIONS II



A slope field plot for the first order linear differential equation $10\frac{dy}{dx} + 3xy + x^3 = 0$

D I F F E R E N T I A L E Q U A T I O N S I I

Lesson 1

Further A-Level Pure Mathematics, Core 2
Differential Equations II

1.1 Separating The Variables

Previously, in Differential Equations I, we looked at first-order differential equations that could be written in the form,

$$\frac{dy}{dx} = f(x)\,g(y)$$

These were solved by separating the variables, in other words, first rewriting the equation in the form,

$$\int \frac{1}{g(y)}\ dy = \int f(x)\ dx$$

The hope now is that both integrations can be done and the resulting algebra manipulated to give an answer in the form,

$$y = f(x)$$

As an example of the method being applied but not quite working out as planned consider solving the following differential equation given that $y = 0$, when $x = 0$,

$$\frac{dy}{dx} = \frac{x^2 - 4}{y^2 + 4}$$

This is of the required form with $f(x) = x^2 - 4$ and $g(y) = \dfrac{1}{y^2 + 4}$ and so,

we separating the variables and proceed as follows,

$$\int y^2 + 4\ dy = \int x^2 - 4\ dx$$

$$\frac{y^3}{3} + 4y = \frac{x^3}{3} - 4x + c$$

Using the information that $y = 0$ when $x = 0$, gives that $c = 0$ and so,

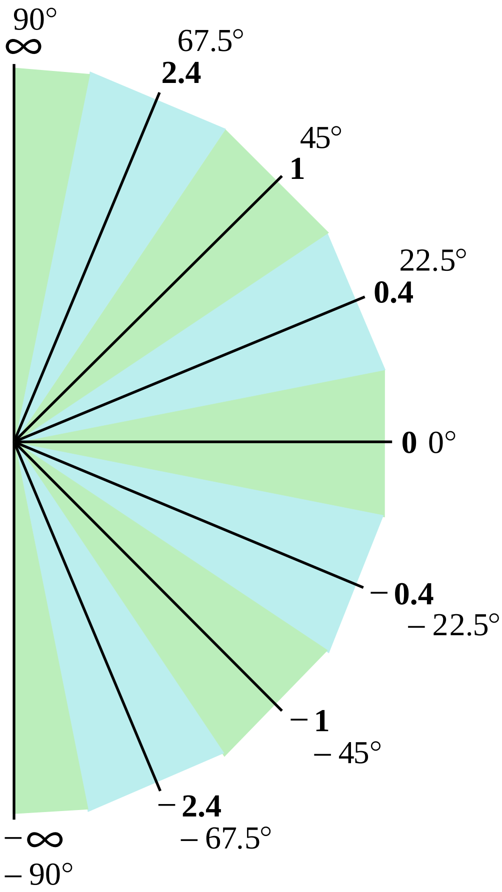
$$x^3 - y^3 = 12(x + y)$$

We've got an answer in that this equation describes the relationship between x and y. However, graphing the solution, or working out another y given an x is not straight forward because the solution is not in the form $y = f(x)$. Nor is it obvious how to get it into that form.

One way around this problem that also gives a bigger picture of the differential equation for other values of the constant c is to produce a slope field plot. This is quite a lot of work by hand; to do it repeatedly software would be used. The idea is to return to the original differential equation and insert a grid of points into it thereby finding out what the gradient is doing at each of those points.

1.2 A Slope Field Plot

When producing a slope field plot by hand, the possible gradients are reduced to eight as indicated by the following diagram. Gradient is intimately linked to the *tan* (x) function so, for example, a slope of 22.5° corresponds to a gradient of $tan(22.5°) = 0.414$ which the diagram approximates with 0.4



Anything sufficiently close to 0.4 is drawn as a slope of 22.5°. So, for example, a gradient of 0.3 is drawn on the slope field plot as a line segment at 22.5°.

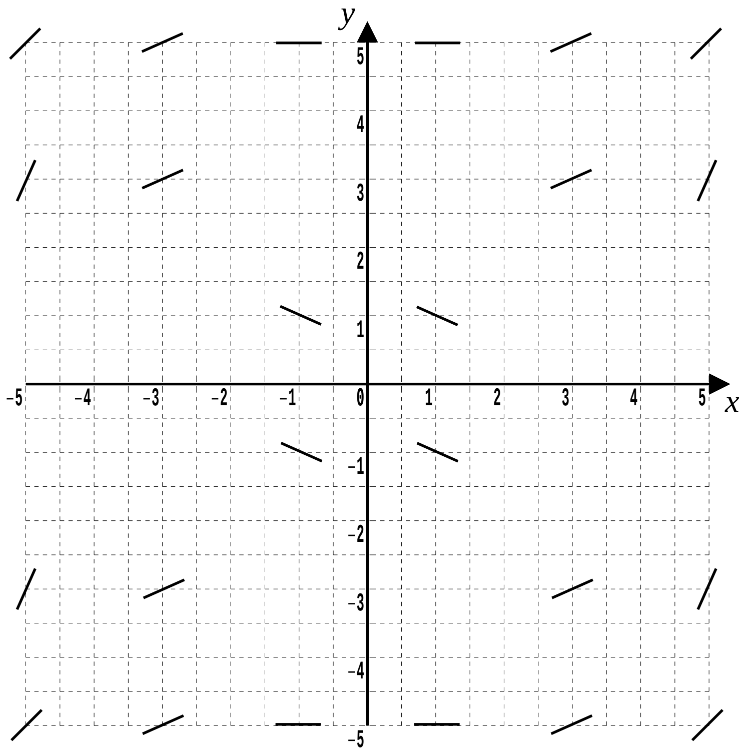
Add in the missing entries in this spreadsheet for the differential equation,

$$\frac{dy}{dx} = \frac{x^2 - 4}{y^2 + 4}$$

		x					
		−5	−3	−1	1	3	5
y	5	0.7	0.2	−0.1	−0.1	0.2	0.7
	3	1.6	0.4			0.4	1.6
	1			−0.6	−0.6		
	−1			−0.6	−0.6		
	−3	1.6	0.4			0.4	1.6
	−5	0.7	0.2	−0.1	−0.1	0.2	0.7

[4 marks]

From the spreadsheet, the slope field plot is drawn.
Add in the slopes corresponding to the entries you added to the spreadsheet.



[4 marks]

We are now all set to sketch the particular solution to the differential equation,

$$\frac{dy}{dx} = \frac{x^2 - 4}{y^2 + 4} \quad \text{for which } y = 0 \text{ when } x = 0$$

using only the slope field plot.

Add the solution curve to the above diagram of the slope field plot.

[2 marks]

Notice that this required no knowledge of the solution. In other words, in producing the (admittedly approximate) plot we did not need to first obtained the entwined relationship between x and y given by,

$$x^3 - y^3 = 12(x + y)$$

If you have access to a graph plotter that can plot this equation you may like to see what the exact curve looks like and so see how good your approximate answer is.

Finally, let's finish with the observation that you could draw the solution curve to the same differential equation but given a different point in the slope field. In fact the slope field plot gives an insightful overview of the possible behaviours of the differential equation.

1.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

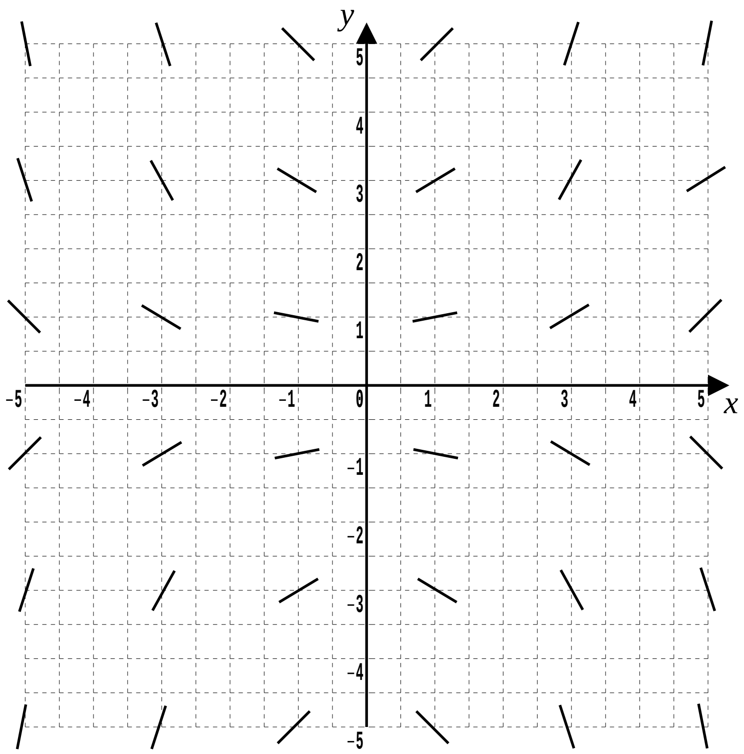
(i) By separating the variables, solve the differential equation,

$$\frac{dy}{dx} = 0.2xy \text{ given that } y = 1 \text{ when } x = 0$$

Present your solution in the form $y = A\,e^{k\,x^2}$ where A and k are constants to be found.

[5 marks]

(ii) Sketch your part (i) solution on this slope field plot.



[2 marks]

Question 2

(i) By separating the variables, solve the differential equation,

$$\frac{dy}{dx} = \frac{x+1}{y+1} \text{ given that } y = -2 \text{ when } x = 0$$

presenting solution(s) in the form $y = f(x)$ the derivation of which may involve completing the square.

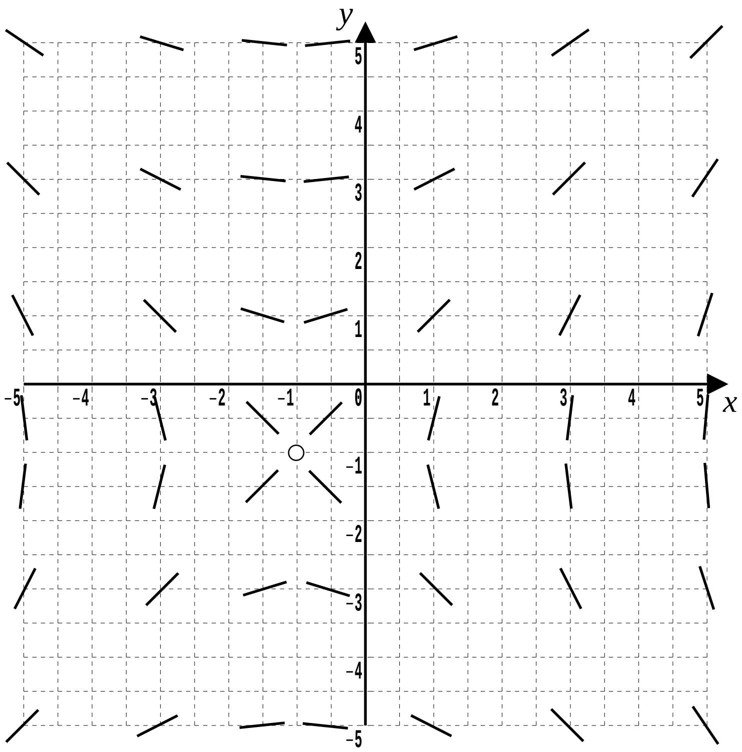
Check for extraneous solutions and remove any found.

[6 marks]

(ii) Here is a slope field plot for the differential equation.

Note the gradient is undefined at $(-1, -1)$.

Graph your part (i) solution(s) on this plot.



[2 marks]

Question 3

Kevin is investigating the first-order linear ordinary differential equation,

$$\frac{dy}{dx} = \frac{4x + y}{x^2 + 1} \qquad \text{given that} \quad y = -2 \quad \text{when} \quad x = 0$$

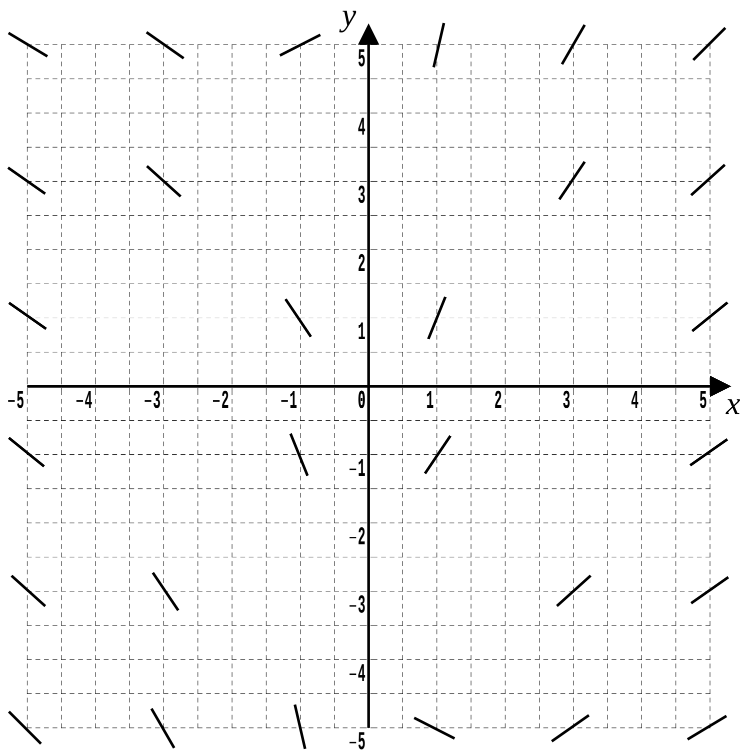
His physics teacher has advised him that this is not solvable using algebra and so he decides to produce a slope field plot to find the approximate solution curve. Here is his partly completed spreadsheet for the slope field plot,

		x					
		− 5	− 3	− 1	1	3	5
y	5	− 0.6	− 0.7	0.5	4.5	1.7	1.0
	3	− 0.7	− 0.9			1.5	0.9
	1	− 0.7		− 1.5	2.5		0.8
	− 1	− 0.8		− 2.5	1.5		0.7
	− 3	− 0.9	− 1.5			0.9	0.7
	− 5	− 1.0	− 1.7	− 4.5	− 0.5	0.7	0.6

(i) Complete Kevin's spreadsheet for him.

[4 marks]

(ii) Add in the slopes on the following slope field plot corresponding to the entries you added to Kevin's spreadsheet.



[4 marks]

(iii) Add the solution curve that passes through the point (0, – 2)

[2 marks]

Question 4

(i) By separating the variables, solve the differential equation,

$$\frac{dy}{dx} = x\,e^{x-y} \text{ given that } y = -3 \text{ when } x = 0$$

Present your answer in the form $y = f(x)$

[4 marks]

(ii) Show that the roots of your part (i) solution curve are given by,

$$x + ln(1 - x) + 3 = 0$$

[4 marks]

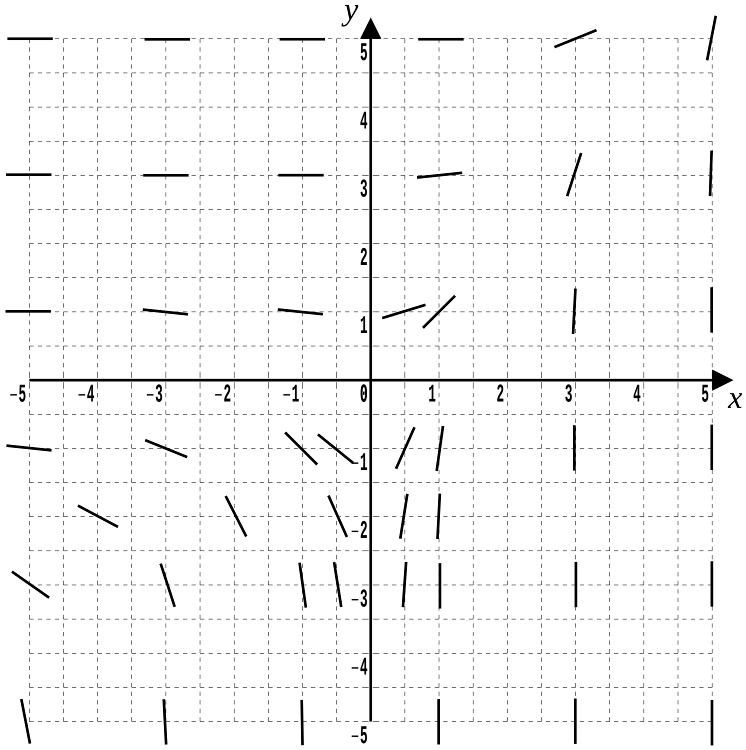
(iii) Explain why any roots of the part (i) solution curve must satisfy $x < 1$

[1 mark]

(iv) Use the Newton-Raphson starting from $x_0 = -5$ to find a root of the part (iii) equation correct to 2 decimal places.

[4 marks]

(v) Here is a slope field plot for the differential equation. Keeping in mind that the gradient is zero on the *y*-axis, graph your part (i) solution.



[2 marks]