Lesson 2

2.1 Differential Equation Terminology

A "first order" differential equation is one in which there are only first derivatives and no derivatives of a higher order.

So, only $\frac{dy}{dx}$ in the equation and no occurrences of, for example, of $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$.

The degree, *n*, is the highest power of the highest derivative.

For example $\left(\frac{d^3y}{dx^3}\right)^4 + 2xy\left(\frac{d^2y}{dx^2}\right)^5 = 1$ is of order 3 and degree 4.

When a differential equation is described as "ordinary" it means that it only has a single independent variable, usually *x*. Typically, *y* then takes on a value that is determined by *x*, making *y* the dependent (upon *x*) variable.

In this lesson the interest is focussed upon first order, ordinary, ***linear***, differential equations. To be described as linear the the differential equation can be manipulated into the form,

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

Crucially, that *y* next to the $P(x)$ is just a *y* and NOT a function of *y*.

To emphasise: there can be no $\sqrt{y}$, $sin(y)$ or y^2 (for example) in the equation.

2.2 Exercise

For each of the following decide if it is a first order ordinary linear differential equation or not. For those that are not, explain why.

(i) $x^2\frac{dy}{dx} + \frac{y}{x} = sin(x)$

(ii) $y\frac{dy}{dx} + 2x = \sqrt{x^2 + 1}$

(iii) $\left(\frac{dy}{dx}\right)^2 + xy = sin(x)$

(iv) $\frac{d^2y}{dx^2} + \sqrt{x}\,y = \frac{1}{x}$

(v) $cos(x)\frac{dy}{dx} + y\,sin(x) = 1$

(vi) $\frac{dy}{dx} = y\,cosh(x)$

[6 marks]

2.3 The Integrating Factor

Differential equations that can be written in the form,

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

are interesting because, although the variables cannot be separated, there is a cunning solution technique. It involves multiplying every term by what is termed the integrating factor, *I*.

$$\text{Integrating Factor,} \quad I = e^{\int P(x)\,dx}$$

The idea is that by multiplying through by the integrating factor, the LHS becomes recognisable as being the answer to a product rule (of differentiation).

2.4 Example #1

To solve $\frac{dy}{dx} + 2xy = e^{-x^2}$ first identify that $P(x) = 2x$ and $Q(x) = e^{-x^2}$

$$\int P(x)\;dx = 2\int x\,dx = x^2$$

If there are no occurrences of the modulus function the constant of integration may be omitted, as done here. This is because when multiplying through by the integrating factor, this constant ends up cancelling with itself. If this short-cut is not to your liking put in the constant. Beware: textbooks typically don't put it in, not even (next lesson) when a situation occurs when perhaps they should !

$$I = e^{\int P(x)\,dx} = e^{x^2}$$

$$\frac{dy}{dx} + 2xy = e^{-x^2} \qquad \text{The differential equation to be solved}$$

$$e^{x^2}\frac{dy}{dx} + 2x\,e^{x^2}y = e^{x^2}e^{-x^2} \qquad \text{Multiplying through by } I$$

Can you spot that the LHS could have come from a product rule differentiation ?

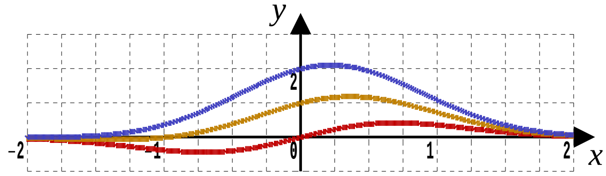
Integrate both sides with respect to *x*

$$\int e^{x^2}\frac{dy}{dx} + 2x\,e^{x^2}y\,dx = \int 1\;dx$$

$$e^{x^2}y = x + c$$

$$y = \frac{x + c}{e^{x^2}} \quad \text{is the general solution}$$

[4 marks]



Solution curves for $\frac{dy}{dx} + 2xy = e^{-x^2}$

for $c = 0$ (red), $c = 1$ (gold) and $c = 2$ (blue)

2.5 Example #2

To solve $\dfrac{dy}{dx} + y\,sin(x) = e^{cos(x)}$ noticing $P(x) = sin(x)$ and $Q(x) = e^{cos(x)}$

$$\int P(x)\ dx = \int sin(x)\ dx = -\,cos(x)$$

If there are no occurrences of the modulus function the constant of integration may be omitted, as done here.

$$I = e^{\int P(x)\,dx} = e^{-\,cos(x)}$$

$$\frac{dy}{dx} + y\,sin(x) = e^{cos(x)}$$

$$e^{-\,cos(x)}\frac{dy}{dx} + sin(x)\ e^{-\,cos(x)}y = e^{cos(x)}\,e^{-\,cos(x)}$$

integrate both sides with respect to x

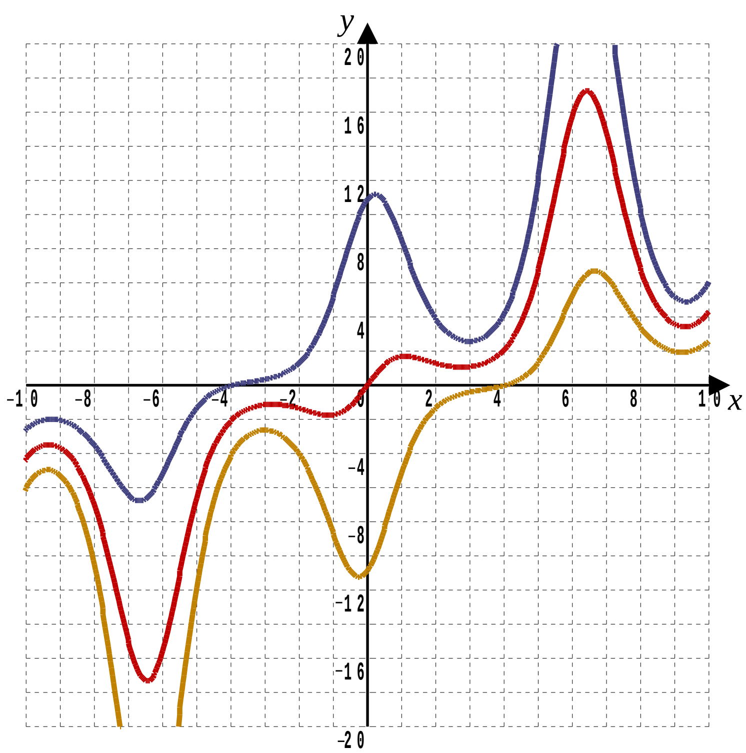
$$\int e^{-\,cos(x)}\frac{dy}{dx} + sin(x)\ e^{-\,cos(x)}y\,dx = \int 1\ dx$$

$$e^{-\,cos(x)}y = x + c \qquad \text{for any real constant } c$$

$$y = \frac{x + c}{e^{-\,cos(x)}}$$

$$y = (\,x + c\,)\,e^{cos(x)} \qquad \text{is the general solution}$$

[4 marks]



Solution curves for $\dfrac{dy}{dx} + y\,sin(x) = e^{cos(x)}$

for c = 0 (red), c = – 4 (gold) and c = 4 (blue)

2.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Find the general solution of the differential equation,

$$\frac{dy}{dx} - 3y = e^x$$

Give an exact answer in the form $y = f(x)$

[5 marks]

Question 2

Find the general solution of the differential equation,

$$\frac{dy}{dx} + y = x$$

Give an exact answer in the form $y = f(x)$

[5 marks]

Question 3

(i) By means of an integrating factor find the general solution to,

$$\sqrt{x}\,\frac{dy}{dx} + y = 1 \qquad x, y \in \mathbb{R},\ x > 0$$

[6 marks]

(ii) This particular equation can also be solved by separating the variables. Show that doing so gets to the same answer.

[6 marks]

Question 4

(i) Find the general solution to the differential equation,

$$\frac{dy}{dx} + 2\,tan(x)\,y = cos^3(x)$$

[7 marks]

Given that $y = 2$ at $x = 0$,

(ii) Find the particular solution which satisfies this condition.

[3 marks]

Question 5

Find the general solution of the differential equation,

$$\frac{dy}{dx} + 2xy = x^3$$

Give an exact answer in the form $y = f(x)$

[8 marks]