Lesson 4

Further A-Level Pure Mathematics, Core 2

Differential Equations II

4.1 More Involved Differential Equations

In general, a second order differential equation is one that contains second derivatives and no higher order derivatives. If it is homogeneous then each term contains the function *y* or one of its derivatives.

So it's of the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad \text{where } a,\ b \text{ and } c \text{ are real valued constants}$$

This differential equation is linear because there is no function applied to the *y* other than it being multiplied by a constant. Such equations are solved by knowing what the format of the solution is. These formats will be revealed shortly but to lessen the feeling that rabbits are being pulled out of hats, we'll first revisit the solving of a first order linear homogeneous equation.

4.2 Solving a 1st Order Linear Homogeneous Differential Equation

The differential equation to be solved is,

$$a\frac{dy}{dx} + by = 0 \qquad \text{where } a \text{ and } b \text{ are real valued constants}$$

$$\frac{dy}{dx} = -\frac{b}{a}y \qquad \text{separating the variables}$$

$$\int \frac{1}{y}\,dy = k\int 1\,dx \qquad \text{where } k = -\frac{b}{a}$$

$$ln|y| = kx + c$$

$$|y| = Be^{kx} \qquad \text{for } B > 0$$

$$y = Ae^{kx} \qquad \text{is the general solution}$$

From $k = -\frac{b}{a}$ we get $ak + b = 0$ which is termed the auxiliary equation

It's interestingly visually similar to where we started from; $a\frac{dy}{dx} + by = 0$

4.3 Solving a 2nd Order Linear Homogeneous Differential Equation

Given an equation in the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \text{where } a,\ b \text{ and } c \text{ are real valued constants}$$

write down the auxiliary equation,

$$a\,m^2 + bm + c = 0$$

The two values of m that make the auxiliary equation true are performing the same roll as the value of k that made $ak + b = 0$ true previously.

Consider the discriminant, D, of the auxiliary equation.

If $\boldsymbol{D > 0}$ there are two real roots, α and β and the differential equation has a general solution,

$$y = A\,e^{\alpha x} + B\,e^{\beta x} \quad \text{for any arbitrary real constants } A \text{ and } B$$

If $\boldsymbol{D = 0}$ there is one repeated real root α and the differential equation has a general solution,

$$y = (A + Bx)\,e^{\alpha x} \quad \text{for any arbitrary real constants } A \text{ and } B$$

If $\boldsymbol{D < 0}$ there are two complex conjugate roots, $\alpha, \beta = p \pm q\mathrm{i}$ and the differential equation has a general solution,

$$y = e^{px}(A\cos(qx) + B\sin(qx)) \quad \text{for any arbitrary real constants } A \text{ and } B$$

4.4 Three Examples (One of each type)

4.4.1 Two Real Roots

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$m^2 + 5m + 6 = 0 \quad \text{the auxiliary equation}$$

$$(m + 2)(m + 3) = 0$$

$$\text{Either } m = -2 \text{ or } m = -3$$

$$y = A\,e^{-2x} + B\,e^{-3x} \quad \text{for any arbitrary real constants } A \text{ and } B$$

4.4.2 One (Repeated) Real Root

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$$

$$m^2 + 10m + 25 = 0 \quad \text{the auxiliary equation}$$

$$(m + 5)(m + 5) = 0$$

$$m = -5$$

$$y = (A + Bx)\,e^{-5x} \quad \text{for any arbitrary real constants } A \text{ and } B$$

4.4.3 Two Complex Conjugate Roots

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

$$m^2 - 4m + 13 = 0 \quad \text{the auxiliary equation}$$

$$m = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 13}}{2}$$

$$= 2 \pm 3\mathrm{i}$$

$$y = e^{2x}(A\cos(3x) + B\sin(3x)) \quad \text{for any arbitrary real constants } A \text{ and } B$$

4.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 34

Question 1

Find the general solution to each of the following differential equations,

(i) $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$$

[3 marks]

(ii) $$16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 0$$

[3 marks]

Question 2

Find the general solution to the differential equation, $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$

[3 marks]

Question 3

(i) Find the general solution to the differential equation,

$$y'' + 4y' + 5y = 0$$

[3 marks]

(ii) Given that $y(0) = 0$ and $y'(0) = 2$ find the particular solution.

[5 marks]

Question 4

Find the general solution to each of the following differential equations,

(i) $15y'' - 7y' - 2y = 0$

[3 marks]

(ii) $4y'' + 20y' + 25y = 0$

[3 marks]

(iii) $y'' + \sqrt{3}\,y' + 3y = 0$

[3 marks]

Question 5

Find the solution to the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ for which $y = 0$ and $\frac{dy}{dx} = 3$ at $x = 0$

[8 marks]