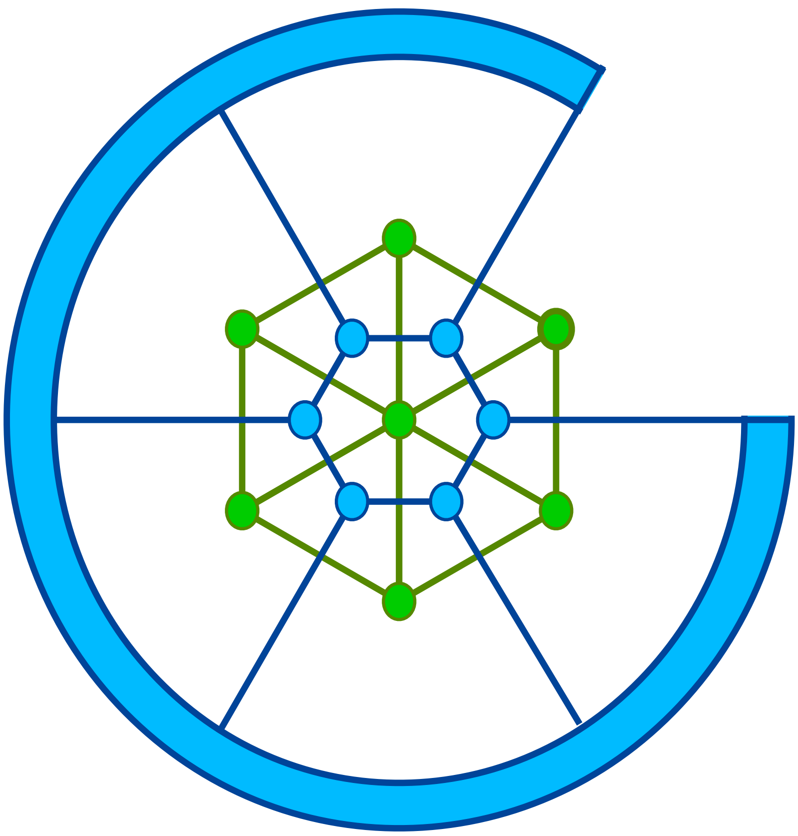
Undergraduate Lectures in Mathematics

Graph Theory I

A Guide for Beginners

Martin Hansen



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Graph Theory I : A Guide for Beginners
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Cover Image

The cover is unconventional in that the graph depicted has a vertex that's been smeared into an arc in order to show in a more symmetrical way a well known graph and its dual. In green is the graph of the wheel, ω_6, and, in blue, its dual.

About this work

These lecture notes are for an third year undergraduate course on Graph Theory. They have a focus on the aspects of the topic that interest me and which, I hope, makes for a lively, colourful and somewhat “different” presentation. Assumed are the basics of Matrix algebra, although key ideas are briefly revisited. I have included comprehensive solutions to the exercises as, to me, a part of what makes mathematics interesting is the fine detail. I enjoy developing and perfecting robust solutions to sets of problems.

In writing these lectures I consulted a wide variety of sources. In particular I have consulted extensively the powerpoint presentations given to undergraduates at the University of Derby by Dr Nicholas Korpelainen, Professor Robin Wilson's classic book, “Introduction to Graph Theory”, now in its fifth edition, alongside the related Open University's MT365 course on the subject. Many of the questions in the exercises are my own, often variations on established themes; several are what might be termed “the classics” of the subject.

Martin Hansen
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January 2023

GRAPH THEORY I

Lecture 1

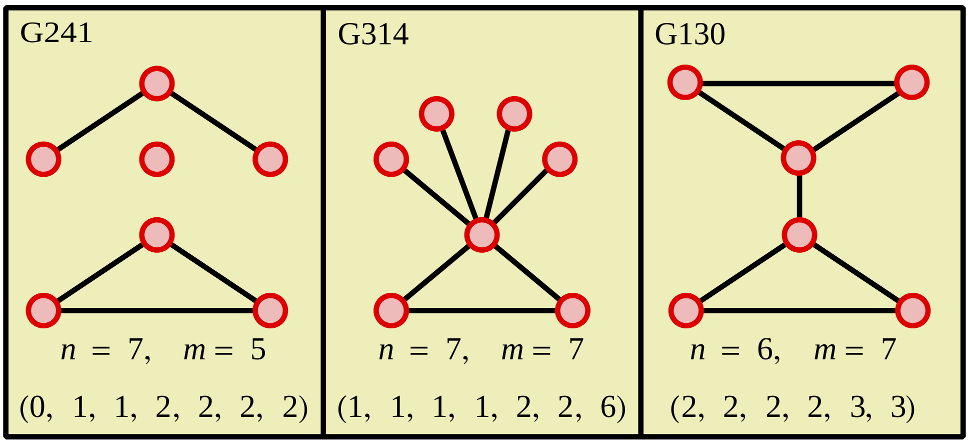
Undergraduate Lectures in Mathematics
A Third Year Course
Graph Theory I

1.1 What is a Graph ?

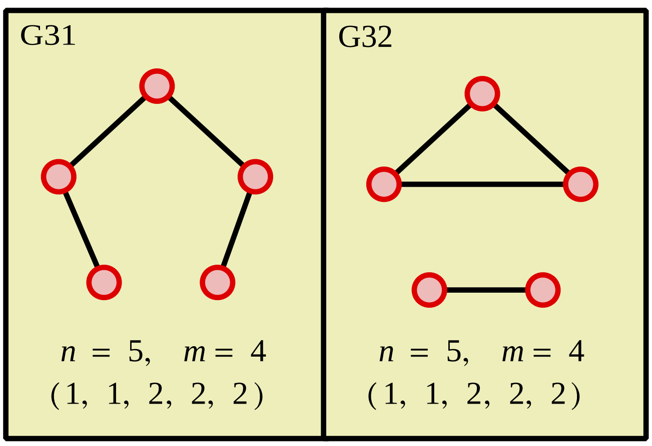
A graph is an ordered pair of sets, $G = (V(G), E(G))$, where the elements of V and E are called *vertices* and *edges* respectively. Each edge is associated with two distinct vertices and is said to join them. As beginners, our interest is in *simple* graphs where, at most, there can be one edge between any vertex pair.

1.2 Visualisation of Graphs

Three examples of graphs are given below. Various catalogues of small simple graphs have been made over the years and the numbers G241, G314 and G130 refer to a catalogue from The Open University. Modern catalogues can be found online in the form of searchable databases. G241 is an example of a graph that is *disconnected*, whilst G314 and G130 are *connected*.

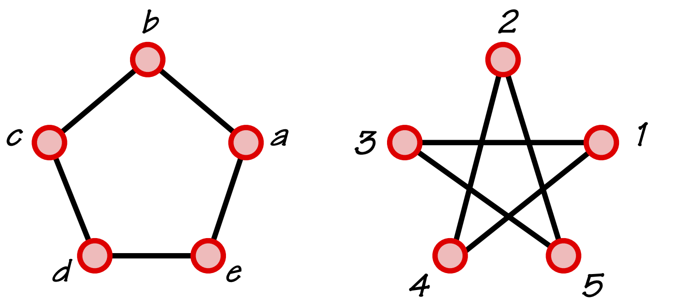


The number of vertices of a graph G is usually denoted by n and is termed the *order* of the graph. The number of edges, m, is the *size* of G. The number of edges that meet at a vertex gives the *degree* of that vertex. The degrees of all the vertices of a graph may be listed as a monotonic increasing sequence, one for which $a_k \leqslant a_{k+1}$ for all positive integer k less than n. Two topologically distinct graphs may have the same *degree sequence* as G31 and G32 demonstrate.



1.3 Some Graph Descriptors

If it is helpful to do so each vertex of a graph may be labelled distinctly, typically using either letters or numbers.

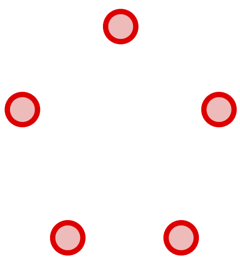


Two labelled graphs G and H are *isomorphic* if there is a bijection f (a one-to-one and onto function) between $V(G)$ and $V(H)$ such that $\{v, w\} \in E(G)$ if and only if $\{f(v), f(w)\} \in E(H)$. Intuitively, isomorphism identifies when two seemingly different graphs, such as the two above, have the same underlying structure. To see that the two graphs above are isomorphic notice that each connects the five vertices as if they are the beads on a loop of string. If required, an isomorphism can be explicitly stated as being, for example, in this case,

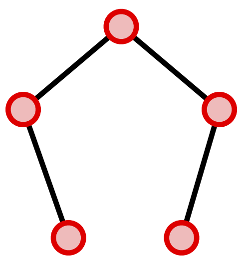
$$f(a) = 1,\ f(b) = 3,\ f(c) = 5,\ f(d) = 2 \text{ and } f(e) = 4$$

Note that "isomorphic to" defines an equivalence relation on any set of labelled graphs. It partitions such a set into equivalence classes, called isomorphism classes. Any catalogue of graphs will only give one example from an isomorphism class.

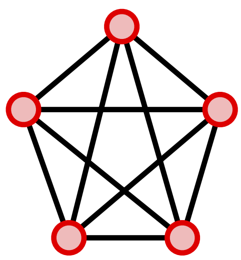
The above two graphs are both a cycle on 5 vertices, C_5. This is an example of a *cycle graph*, a graph that consists of a single cycle of vertices and edges and denoted C_n, $n \geqslant 3$. The graph C_5 is also an example of a *regular graph*, one where all the vertices have the same degree, in this case 2. It could be described as being 2-regular. A regular graph is *complete* if each vertex is joined to each of the others by exactly one edge, and denoted K_n. A regular graph with no edges is termed a null graph, N_n. A path on $n > 2$ distinct vertices is written P_n, and will have two end vertices of degree at least one and thread its way through each of the remaining $(n - 2)$ vertices which will each be of degree at least 2.



N_5



P_5



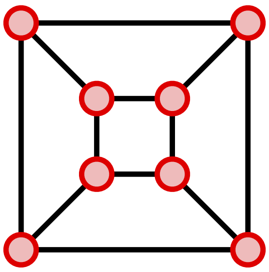
K_5

1.4 Polyhedra

Many everyday objects, polyhedral in shape, can be modelled straight forwardly as graphs. The photographs below are of a waste paper basket. On the left the basket is upside down, showing that it is, topologically speaking, equivalent to a cube. On the the right it is photographed directly from above.

Photographs by Martin Hansen

Of the two views, the photograph to the right translates most easily into a graph, and the result is given below.



What makes this graph the best representation of a cube is the fact that it is uncluttered to look at and this is due in part to the fact that it is a *planar* graph; it can be presented without edges being drawn over one another. The edges of the graph divide the plane into regions called the *faces* of the plane graph. These regions are all bounded but for one, that one being termed the *infinite* or *external* face. The *length* of a face is the number of edges bounding it.

Theorem 1.1 : Euler's Polyhedral Formula

If a connected plane graph has n vertices, m edges and f faces then,

$$n - m + f = 2$$

1.5 Reasoning Why Euler's Polyhedral Formula is True

For the cube, the associated planar graph has 8 vertices, 12 edges and 6 faces, and Euler's polyhedral formula holds in this case. More than simply observing that it holds for a cube, we can look into why it holds. We'll do this in a way that will readily generalise to show that the same formula it holds for all polyhedra that do not have any holes in them.

The key idea is to work with the cube's planar graph and identify some operations that reduce the complexity of that graph whilst leaving invariant the value of $n - m + f$.

1.5.1 The Triangularisation Move

Pick a face, if any, with more than three sides. Add an edge across that face. Notice that this adds one edge and one face to the graph but leaves the value of $n - m + f$ unchanged because,

$$n - (m + 1) + (f + 1) = n - m + f$$

1.5.2 Face Removal Move

Look for a face, if any, which shares precisely one edge with the exterior face. Remove this face by removing the one shared edge. Notice that this subtracts one edge and one face from the graph but does not alter the value of $n - m + f$ because,

$$n - (m - 1) + (f - 1) = n - m + f$$

1.5.3 Vertex Removal Move

Look for a face, if any, which shares precisely two edges with the exterior face. Remove this face by removing both these shared edges and their shared vertex. Notice that this subtracts two edges, one vertex and one face from the graph. For the new graph we have,

$$(n - 1) - (m - 2) + (f - 1) = n - m + f$$

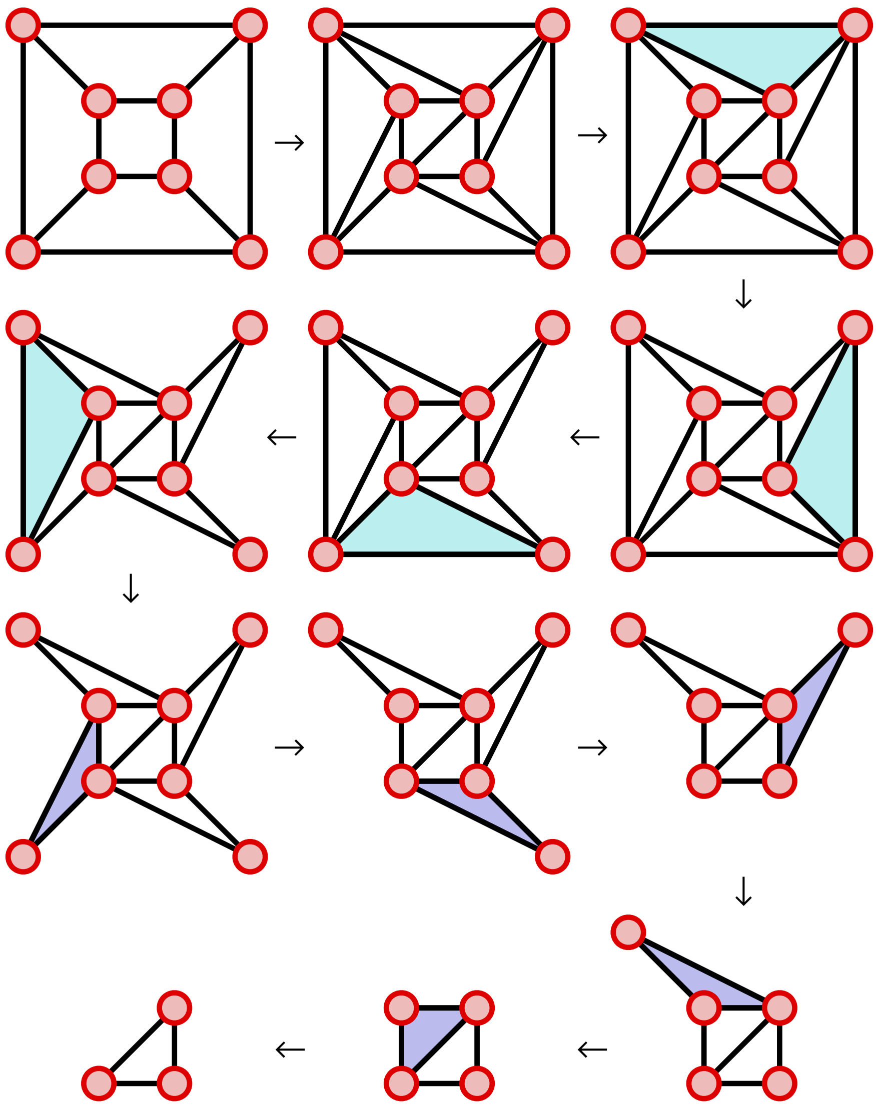
showing that, once again, the value of $n - m + f$ is unaltered by the move.

1.5.4 Applying the Moves

Apply the triangularisation move repeatedly until the entire graph has been triangularised. Always apply the vertex removal rule if it is possible to do so (To prevent the graph fragmenting and becoming disconnected). When the vertex removal rule cannot be performed apply the face removal rule once before switching back to always applying the vertex removal rule if it is possible to do so. Continue in this manner until it becomes impossible to do so.

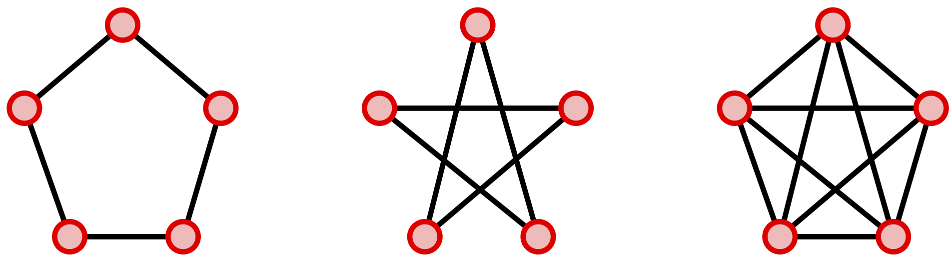
The algorithm will eventually stop with the terminating graph being that of a single triangle for which $n = 3$, $m = 3$ and $f = 2$ (don't forget the external face). For this triangle, $n - m + f = 2$, and so this must also have been true for the original planar graph and associated polyhedron.

The next diagram shows the process being applied to the planar graph associated with a cube.



1.6 The Complement of a Graph

The complement of a graph G is a graph $\overline{G}$ with the same vertex set V but whose edge set E consists of the edges not present in G. The graph sum $G + \overline{G}$ on graph of degree n is therefore the complete graph K_n.
A graph G is *self-complementary* if $G = \overline{G}$



C_5 + $\overline{C_5}$ = K_5

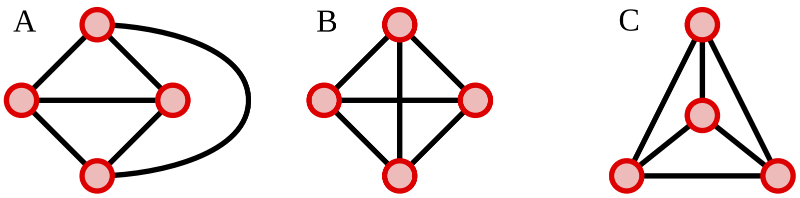
As an example, C_5 and its complement $\overline{C_5}$ are shown. Their graph sum is K_5. Previously it was noted that the two graphs being summed were isomorphic. Thus C_5 is an example of a self-complementary graph.

1.7 Exercise

Marks Available : 80

Question 1

The following three graphs are three representations of the complete graph K_4



(i) Write down the degree sequence for K_4

[1 mark]

(ii) Which one of the three graphs is not planar ?

[1 mark]

(iii) K_4 is an example of a regular graph.
Is it 3-regular, 4-regular or 6-regular ?

[1 mark]

(iv) How many faces has graph C ?

[1 mark]

(v) Which of the five platonic solids could be represented by K_4 ?

[1 mark]

Question 2

Draw all eleven possible unlabelled simple graphs with four vertices.
Under each write its degree sequence.

[6 marks]

Question 3

(i) Provide a proof of the following theorem,

Theorem 1.2 : The Handshaking Lemma
Suppose that a graph G has n vertices and m edges, with degree sequence given by $(d_1, d_2, ..., d_n)$. Then $\sum_1^n d_i = 2m$

[4 marks]

(ii) Explain why The Handshaking Lemma is so called.

[2 marks]

(iii) Using The Handshaking Lemma, prove by contradiction that;

Lemma 1.1 : The Even Number of Odds
Every graph has an even number of odd-degree vertices.

[5 marks]

Question 4

A connected graph is one in which there is a path from any point to any other point on the graph. A graph that is not connected is said to be disconnected. Connected graphs arise naturally when derived from a polyhedron.

(i) Let G be a 3-regular connected planar graph with 20 vertices.
Determine the number of regions (faces) in the graph.

[4 marks]

(ii) Let G be a p-regular connected planar graph with n vertices.
Prove that either the number of vertices or the regularity must be even.

[4 marks]

Question 5

Prove by induction;

Lemma 1.2 : Edge Count of a Complete Graph (A Clique)

The complete graph K_n has exactly $\dfrac{n(n-1)}{2}$ edges

[6 marks]

Question 6

(i) Prove that there are no 3-regular graphs with five vertices.

[3 marks]

(ii) Prove that, if n and r are both odd, then there are no r-regular graphs with n vertices.

[3 marks]

Question 7

Prove the following lemma;

Lemma 1.3 : Vertex Pair Of Equal Degree

In any finite simple graph with at least two vertices, there must be at least two vertices which have the same degree.

[4 marks]

Question 8

Let G be a graph with nine vertices such that each vertex is of either degree 5 or of degree 6. Show that G has at least six vertices of degree 5, or at least five vertices of degree 6.

[4 marks]

Question 9

Prove that, to be disconnected, a graph on n vertices can have, at most, a number of edges, m, that is given by,

$$m = \frac{(n-1)(n-2)}{2}$$

[6 marks]

Question 10

(i) Prove that a set of x elements has 2^x subsets.

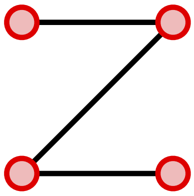
[3 marks]

(ii) Show that there are exactly $2^{\frac{1}{2}n(n-1)}$ labelled simple graphs on n vertices.

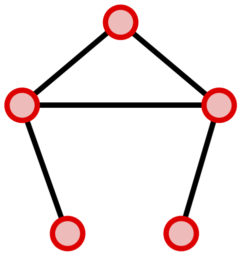
[3 marks]

Question 11

(i) Draw the complements to each of the following graphs.
Under each write its degree sequence.



(1, 1, 2, 2)



(1, 1, 2, 3, 3)

[2 marks]

(ii) A graph G on n vertices is self-complementary with degree sequence,

$$(d_1, d_2, ..., d_n)$$

Determine the relationship between d_i and d_{n+1-i}

[2 marks]

(iii) Use your part (ii) relationship to generate a list of the degree sequences, one of which any self-complementary graphs on five vertices must satisfy.

[4 marks]

(iv) Show that, if a graph G is isomorphic to its complement, then the number of vertices of G has the form $4k$ or $4k + 1$ for $k \in \mathbb{Z}^+$

[5 marks]

(v) Furthermore, if G is regular, show that $n \equiv 1 \pmod 4$

[5 marks]