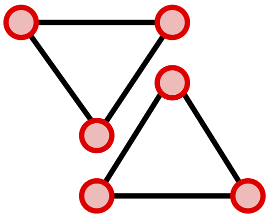
3.11 Answers to 3.10 Exercise

Answer 1

Charles forgot to state that the graph must be connected for his statement to be true. The counterexample to his statement without the word connected is;



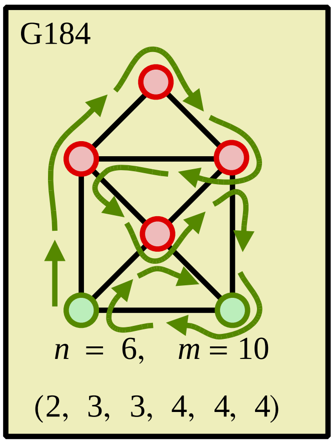
[2 marks]

Answer 2

(i) The degree sequence of G184, given in the diagram, is (2, 3, 3, 4, 4, 4). By Theorem 3.4, “a connected graph is Eularian if and only if each vertex has even degree” and so the odd numbers in the degree sequence mean G184 is not Eularian.

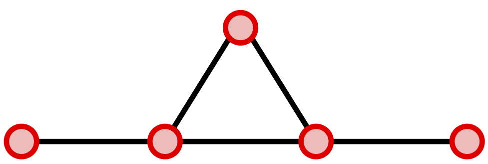
[1 mark]

(ii) The trail will have to start at one of the vertices of odd degree and finish at the other vertex of odd degree, and pass along each and every edge once. One such is depicted below but there are others.



[2 marks]

Answer 3



[2 marks]

Answer 4

Lemma 3.5 : Must Contain A Cycle

Any finite, connected graph, with more than two vertices, in which every vertex is of degree of at least 2, must contain a cycle.

Proof (from "Introduction to Graph Theory by Robin J Wilson")

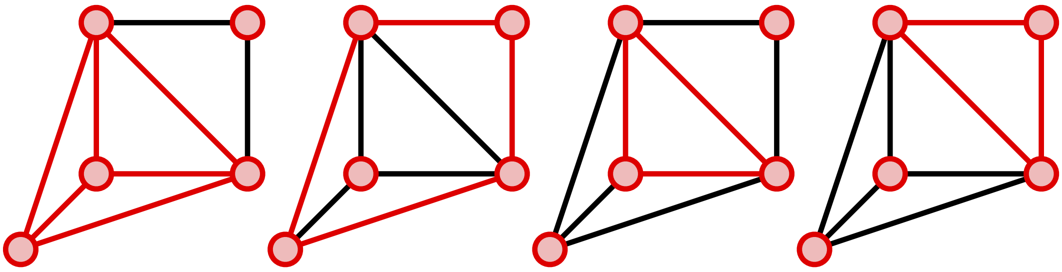
We begin by assuming that the graph, G, is simple.

(If it were not, with loops or multiple edges, the result is trivial)

Let v be any vertex of G and construct a walk v, v_1, v_2, ... inductively, by choosing v_1 to be any vertex adjacent to v and, for each $k > 1$, choosing v_{k+1} to be any vertex adjacent to v_k except v_{k-1}, the existence of such a vertex guaranteed by the hypothesis. Since G has only finitely many vertices, eventually a vertex will be chosen that has been chosen before. Let v_c be the first such vertex in which case the part of the walk that lies between the two occurrences of v_c is the required cycle. □

[4 marks]

Answer 5



maximal clique   not a clique   non-maximal clique   maximal clique

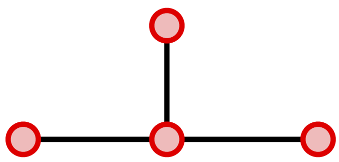
[4 marks]

Answer 6

Theorem 3.1 points out that "A connected graph is Eularian if and only if each vertex has even degree". To have a Eularian Circuit the graph has to first be Eularian. When n is even, K_n is ($n - 1$)-regular, and so all vertices are of odd degree. Thus there can be no Eularian Circuit when n is even.

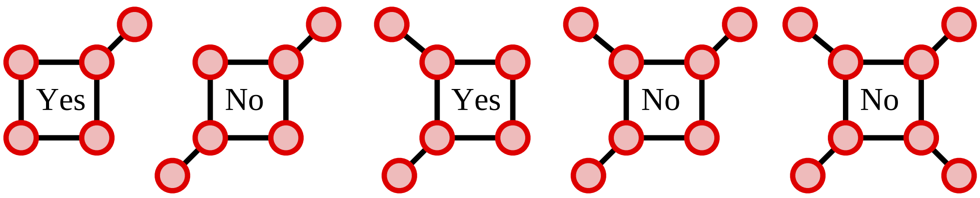
[3 marks]

Answer 7



[2 marks]

Answer 8



[3 marks]

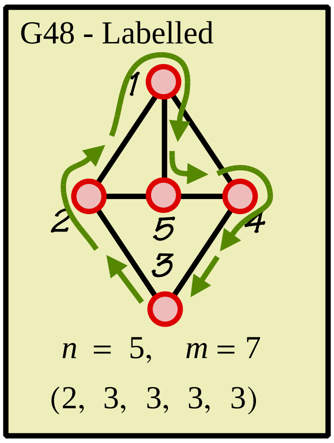
Answer 9

The non-adjacent vertices with their degree sum are,

13 : 3 + 2 = 5

24 : 3 + 3 = 6

35 : 2 + 3 = 5

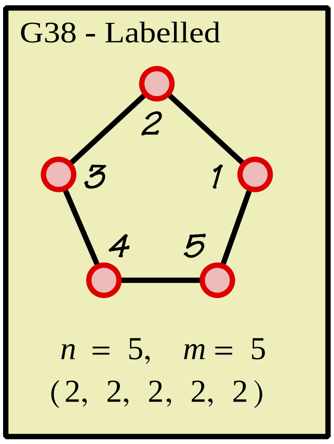


Ore's Theorem requires that all the degree sums are at least 5 which is so.
Thus G48 is Hamiltonian.
The annotation shows one of the two possible Hamiltonian cycles.

[3 marks]

Answer 10

With the C_5 graph labelled as shown, the non-adjacent vertices with their degree sum are,



13 : 2 + 2 = 4

14 : 2 + 2 = 4

24 : 2 + 2 = 4

25 : 2 + 2 = 4

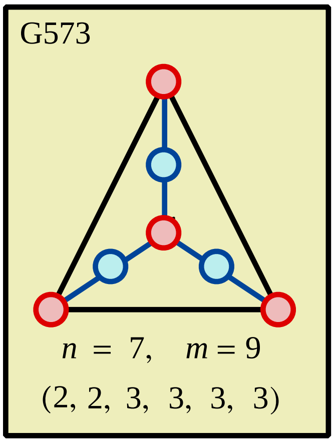
35 : 2 + 2 = 4

Ore's theorem is not satisfied because it requires that all of the degree sums must be at least 5 (the number of vertices) which is not so (none of them are).
In spite of this the graph is obviously Hamiltonian.
We say that Ore's Theorem is a sufficient condition but not a necessary one.

[3 marks]

Answer 11

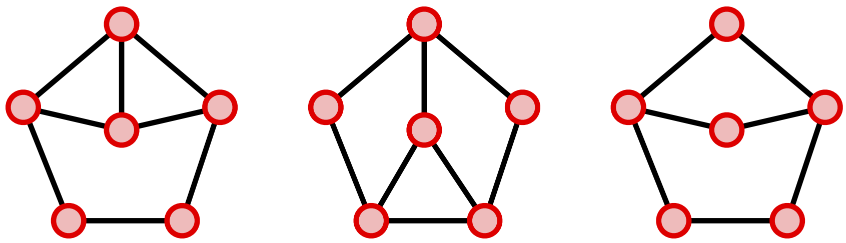
Any vertex of degree two has to be on the Hamiltonian cycle and the edges incident to such a vertex must be traversed. The vertices of degree two and their incident edges are coloured blue in the following annotation of G573,



All three edges that are incident to the centre vertex thus would have to be on any Hamiltonian cycle which means that such a cycle cannot exist.

[3 marks]

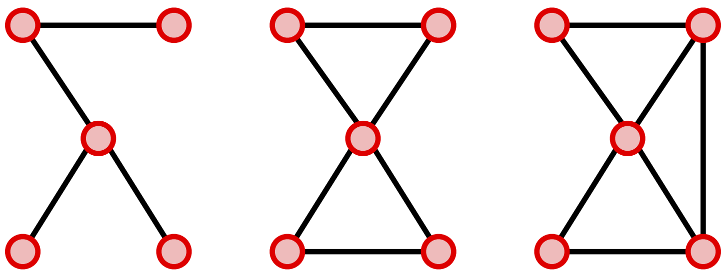
Answer 12



The leftmost graph is Hamiltonian but not unique as there are two distinct Hamiltonian cycles that can be drawn, each missing out a different edge.
The central graph is uniquely Hamiltonian, the cycle missing out the vertical edge.
The rightmost graph is not Hamiltonian at all and so not uniquely Hamiltonian.

[3 marks]

Answer 13



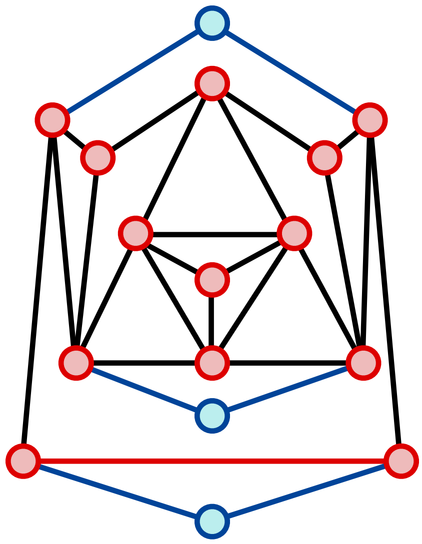
The leftmost graph is neither Eularian nor Hamiltonian.
The central graph is Eularian but not Hamiltonian.
The rightmost graph is not Eularian but it is (uniquely) Hamiltonian.

[3 marks]

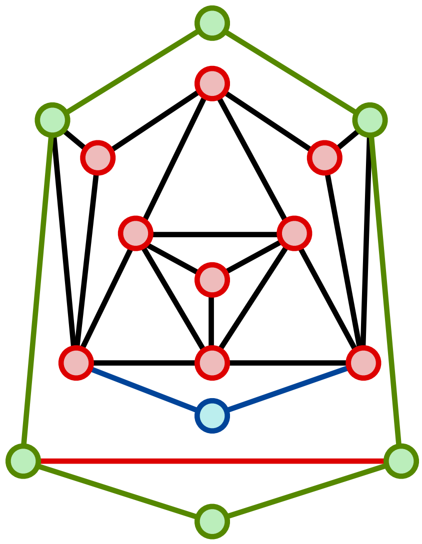
Answer 14

(i) By Theorem 3.1, “A connected graph is Eularian if and only if each vertex has even degree”. Many of the vertices in the graph are of odd degree and so it is not Eulerian.

(ii) Any vertex of degree two has to be on the Hamiltonian cycle and the edges incident to such a vertex must be traversed. The vertices of degree two and their incident edges are coloured blue in the following annotation the given graph,



Of these it is the lowest vertex that is problematical, and results in the edge highlighted in red being excluded from any Hamiltonian cycle. This in turn means that the two side edges must be in the cycle. However, we that have the six cycle highlighted in green below as having to be a part of the Hamiltonian path.

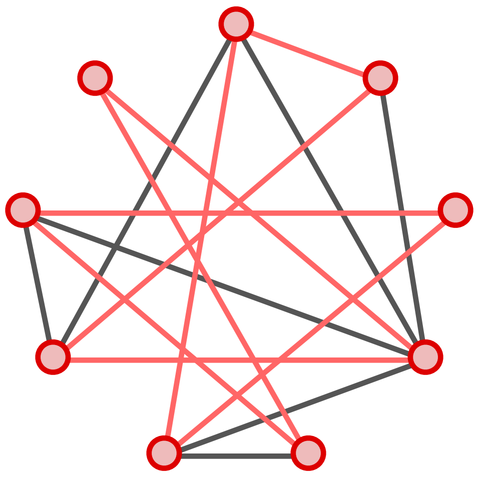


However, on looking at the way this six cycle joins to the rest of the graph it becomes clear that it cannot be a part of a Hamiltonian cycle. Thus the graph is not Hamiltonian.

[4 marks]

Answer 15

The graph needs searching to see if it has a Hamiltonian cycle.
It does !



[4 marks]

Answer 16

Corollary 3.1 : Dirac's Theorem

If G is a simple connected graph with n vertices, where $n \geqslant 3$, and $deg(v) \geqslant \frac{n}{2}$ for each vertex v, then G is Hamiltonian.

Proof

Suppose that we have a graph that satisfies the conditions of the theorem and which is not complete, for if it were, it would be Hamiltonian and we'd be done. Take two non-adjacent vertices, u and v, from the graph G that are not joined by an edge. Consider the sum of the degrees of these vertices, $deg(u) + deg(v)$.

From the theorem's conditions we know that $deg(u) \geqslant \frac{n}{2}$ and $deg(v) \geqslant \frac{n}{2}$.

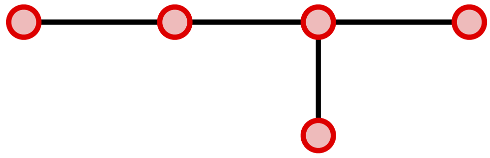
Then $deg(u) + deg(v) \geqslant \frac{n}{2} + \frac{n}{2}$ and we have that, $deg(u) + deg(v) \geqslant n$.

From Ore's Theorem, G is Hamiltonian. □

[4 marks]

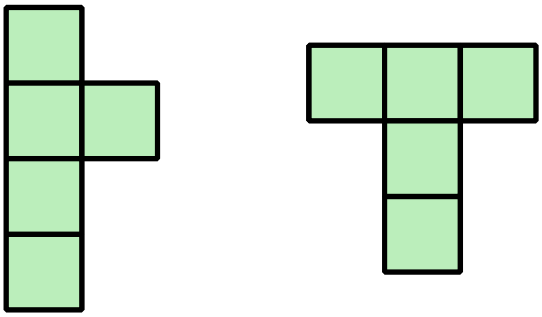
Answer 17

(i) Anything isomorphic to,



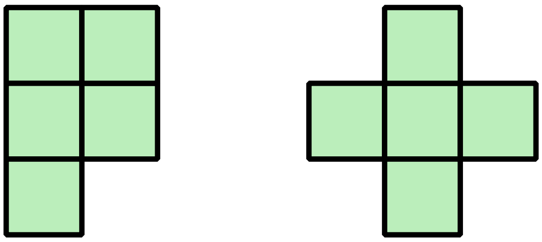
[1 mark]

(ii) Either one of the following two pentominos,



[1 mark]

(iii)



[3 marks]

Answer 18

Lemma 3.6 : Semi-Eularian IFF
A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Proof

(⇒)

If G is semi-Eulerian then there is an open Euler trail, T, in G. Suppose the trail begins at v_1 and end at v_n. Except for the initial occurrence of v_1 and the concluding occurrence of v_n, each time a vertex is encountered, it accounts for two edges adjacent to that vertex, the one before it in the trail and the one after. T uses every edge exactly once. So every edge is accounted for without repetition. In conclusion, the degree of every vertex must be even except for v_1 and v_n which must both be odd.

(⇐)

Suppose u and v are the two vertices of odd degree. Consider the related graph G' where a single edge has been added to G between u and v. Every vertex in G' is of even degree and so by Theorem 3.1, “A connected graph is Eularian if and only if each vertex has even degree”, G' has a closed Euler trail. This closed trail must use the edge between u and v. Thus there must be an open Euler trail in G when the edge between u and v is removed.

□