Shrewsbury School Mathematics Faculty
Arnold Hagger
Mathematics Prize, 1996

Answer as many questions as possible, but aim for complete solutions. Full solutions to a small number of questions is worth more than lots of part answers. The questions are ***not*** *of equal length or difficulty. You may not use a calculator.*

1. Water is being poured into a container, which is of constant cross-section and one metre high. The water is being poured in at a constant rate, but there is a hole in the side of the tank which, when the water is at or above the level of the hole, lets water out at a constant rate. The hole is at a height somewhere between a quarter and three quarters of a metre above the base.

   The container is a quarter full after 2½ minutes, three quarters full after 8½ minutes and full after 12 minutes and 40 seconds. Calculate the height of the hole and how long the container would take to fill if there was no hole.

2. If the times for the first two finishers in the Olympic 100 m final this summer are 9.96 seconds and 9.92 seconds, estimate the distance between them at the finish.

3. A grateful pupil decides to fly me to an exotic location for my birthday. I don't know where I am going, but get on a plane at London at 4pm on Friday evening, and arrive at the Mystery City at 6:30am the following morning. After an exhausting weekend of holiday-ing, I leave the Mystery City at 4pm on Sunday night, arriving back at London at 10:30pm the same evening. All times are local times and are not subject to any strange adjustment (such as 'Daylight Saving Time' or such). Assuming both planes fly at the same constant speed, calculate the longitude of the Mystery City.

4. I have Shrewbury School Mathematics Prize papers, in various forms, going back to 1890. Here is a question from 1890:

   In a boat race 1 mile in length, A crew beat B crew by 10 yards rowing downstream. The A crew beat C crew by 60 yards also rowing downstream. If it takes 1¾ times longer to row upstream than to row downstream, by how much would B crew beat C crew if the race is rowed upstream?

   [There are 1760 yards in a mile].

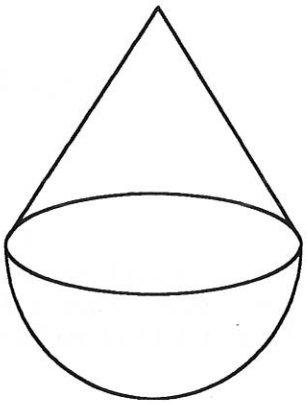
5. Solve for x, y, z and w the simultaneous equations:
$$\begin{aligned} x+y+z &= a \\ x+y+w &= b \\ x+z+w &= c \\ y+z+w &= d \end{aligned}$$

[Turn Over]

6. An ant is at the top corner of a cuboid room which is 5 metres by 6 metres by 7 metres. The only other item in the room is a grain of sugar in the furthest corner from the ant. Calculate the minimum distance the ant must walk to get to the grain of sugar.

7. Prove that for all real numbers a, b and c, $a^2 + b^2 + c^2 - ab - ac - bc \geq 0$.

8. A 'weeble' is made up of a hemisphere and a right cone, as illustrated in the sketch. If the hemisphere is of radius r, and the weeble is made of a uniform material, calculate the maximum possible height of the cone for which the weeble is still stable (meaning the weeble will not fall over when sitting on the hemisphere).



[You may use the facts that the centre of gravity of a cone is ¼ of the way to the top from the base, and the centre of gravity of a hemisphere is ⅜ of the radius from the centre of the circular face].

9. Brian's Casino offers the following game. For a certain fee, the customer is given a fair coin, and he tosses it until he gets a head. If he gets it on the first toss he wins £1; if he gets in on the second toss he wins £2; on the third toss, £4; on the fourth, £8 and so on. Show that *whatever* Brian charges for the game, he is being stupid.

10. Express 1996 in the form $a^2 - b^2$ where a and b are both three digit integers. Explain why this is possible for all leap years for which I have copies of Shrewsbury School Mathematics Prize papers.

11. Explain why the probability of a boy in the school having 29th. February as his birthday depends on when you ask the question. Show how you would go about working out this probability (you are not expected to get a final answer without a calculator) if I asked the question of you today.