**Shrewsbury School**

# The Arnold Hagger Mathematics Prize

**Wednesday January 25th 2006**

Unlike GCSE and, to some extent "A" level, answers are marked largely on *elegance* and *mathematical coherence*. **Try to justify everything you say, try to reason.** Short, beautiful answers will appeal to your examiner more than long, contorted ones.

You are unlikely to score well if you write down only the answers.

The word ***Prove*** has a special significance in the hearts of mathematicians

Although the questions are in roughly ascending order of difficulty you may find you can score more rapidly by skipping certain questions out. In short, pick and choose.

None of these questions depend on anything more than GCSE (though they are, of course, more difficult)

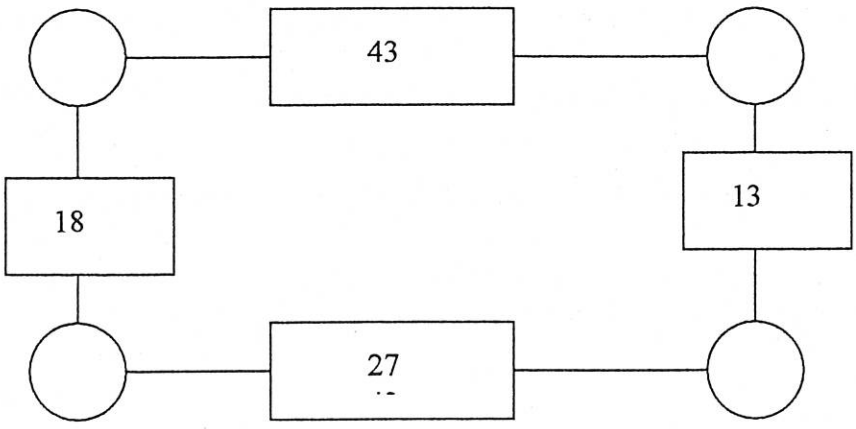
You may wish to pick and choose the questions you answer

1) If 5 men can dig 5 holes in 3 days, how many holes can 4 men dig in 12 days?

2) Find x if $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-4}{x-5}-\frac{x-5}{x-6}$ (from a textbook for 14yo students from 1902)

3) If $m_1a^2+m_2b^2=m_1c^2+m_2d^2$ and $m_1a+m_2b=m_1c+m_2d$ find relationships between a,b,c and d

4) In the diagram below, the numbers in the square boxes are the sum of the numbers in the connected (with a line) circles. Find the numbers in the circles.




5) Say how many and which (if any) of the following 10 statements is/are true, giving reasons :

1. Exactly ONE statement on this list is true

2. Exactly TWO statements on this list are true
3. Exactly THREE statements on this list are true
4. Exactly FOUR statements on this list are true
5. Exactly FIVE statements on this list are true
6. Exactly SIX statements on this list are true
7. Exactly SEVEN statements on this list are true
8. Exactly EIGHT statements on this list are true
9. Exactly NINE statements on this list are true
10. Exactly TEN statements on this list are true

6) I have a 3 litre jug, and a 5 litre jug. Can I measure exactly 7 litres of water?

7) Find the *exact* value of

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}$$

8) One sleepless night I take my family silver out to bury it on a rectangular plot of farmland. I bury it (underground) exactly 21000m from one corner, 18000 metres from the opposite corner, and 6000m from a third corner. How far is it from the fourth corner?

9) Cryptography is the science and study of secret writing. One system is to substitute letters in a *plaintext string* for others. Under the code

C = (qwerty)(uiop)(asd)(fgh)(jk)(lz)(xcv)(bnm),

for instance, each letter is replaced by its successor – the ones at the end of the brackets are replaced by the one at the start, wrapping round.

The string THISQUESTIONISEASY becomes YFODWIRDYOPMODRSDQ, and we write

C("THISQUESTIONISEASY") = "YFODWIRDYOPMODRSDQ"

The bracketed expression above is called a *representation* of the code. If we apply the code C, followed by the code D (doubly enciphering), we'd say this was DC

(because D(C(x)) = DC(x) )

(i) Write down the representation of the code $C^2$.
(ii) Write down the representation of a code which leaves every letter unchanged.
(iii) Write down the representation of $C^{-1}$ – the code which decodes C
(iv) Is there a power of C for which the code leaves every plaintext string unchanged? If so, what is it?

(v) Given two codes, C and D, is the statement

$$CD = DC$$

always true?

10) It is well known that $\sqrt{2}$ cannot be written as a fraction (i.e. $\frac{\text{whole number}}{\text{whole number}}$).

Taking this as known, ***prove*** that $\frac{\sqrt{2}+3}{\sqrt{2}-1}$ is also irrational

11) Determine whether the following are true or false. If true, prove. If false, provide a counter example. In these, p is a prime number. "divides" means divides into exactly with no remainder.

(i) if p divides $a^3$, then p divides a
(ii) if p divides a and p divides $(a^2+b^2)$ then p divides b
(iii) if $a^3$ divides $b^2$ then a divides b
(iv) if p is a prime and p divides $(a^2+b^2)$ and p divides $(b^2+c^2)$ then p divides $(a^2+c^2)$
(v) If m divides $(a^2-1)$ then m divides $(a^4-1)$

12) On each of n straight lines, m points are taken such that no other straight lines can be drawn through any three of the mn points. How many triangles can be made with these points as vertices?

13) How many permutations are there of the first 5 letters of the alphabet with the two vowels not adjacent? The letters cannot be used more than once.

Good Luck!