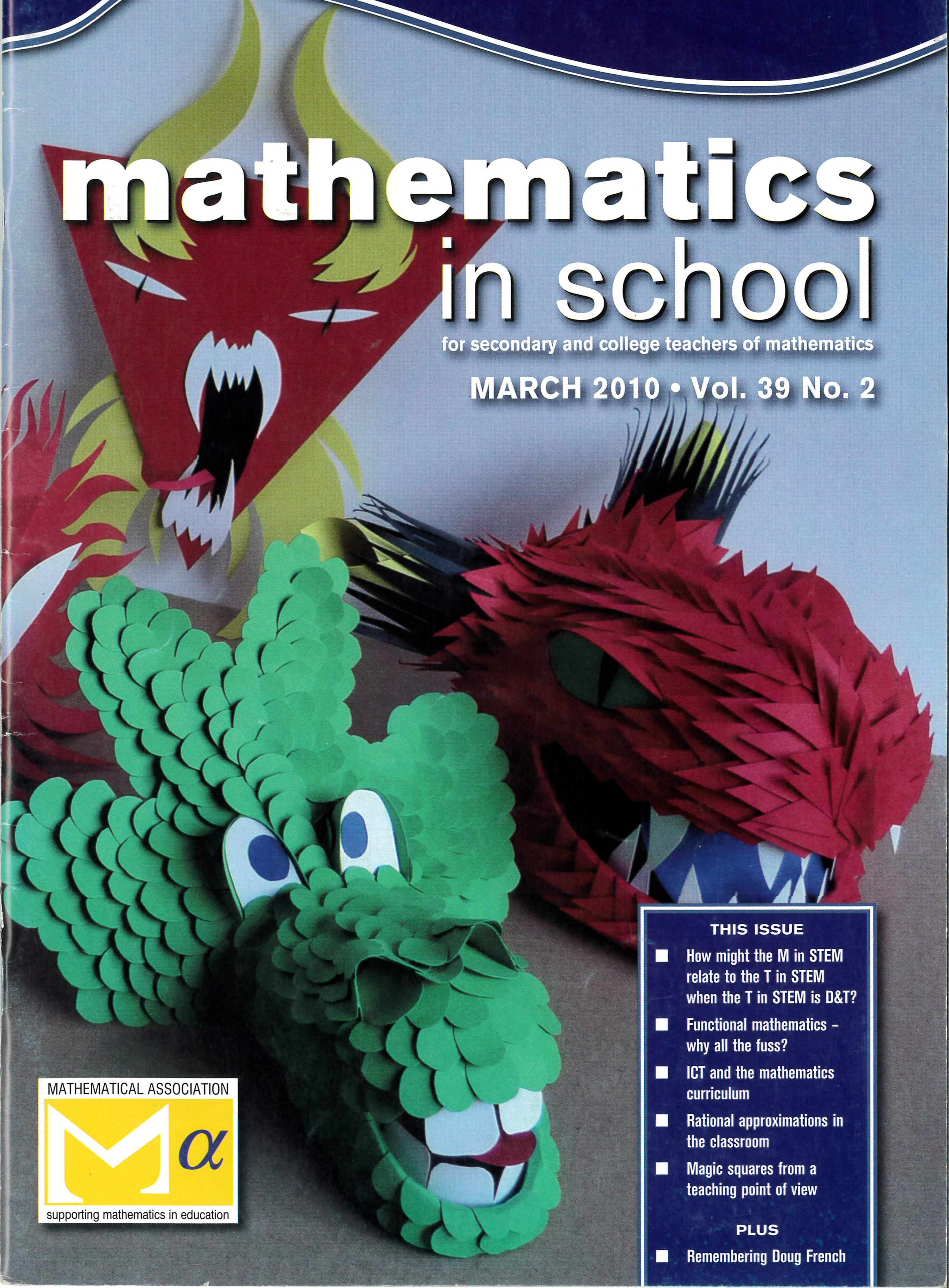
mathematics in school

for secondary and college teachers of mathematics

MARCH 2010 • Vol. 39 No. 2



THIS ISSUE

- How might the M in STEM relate to the T in STEM when the T in STEM is D&T?
- Functional mathematics – why all the fuss?
- ICT and the mathematics curriculum
- Rational approximations in the classroom
- Magic squares from a teaching point of view

PLUS

- Remembering Doug French

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Magic Squares
from a Teaching Point of View
Part 1

by Martin Hansen

The recent article in *Mathematics in School* by Mike Rose rekindled my interest in magic squares (Rose, 2009). What I liked about Mike's investigation was that it focussed on the smallest possible size of magic square, those of order three. It's all too tempting to move swiftly on to bigger magic squares without properly appreciating the subtleties of the 3×3s.

I tried out Mike's investigation on my 11-year-old daughter. To my surprise, I discovered that she did not know what a magic square was. In a one-to-one situation this was quickly remedied but it was clear that before I could use the investigation with my classes I needed to give some preparatory teaching. The resulting lessons are the backbone of this series of articles.

The iconic 3×3 magic square is that known in China as the Lo-shu (Fig. 1).

2	9	4
7	5	3
6	1	8

Fig. 1

In the classroom I start by placing the Lo-shu on a whiteboard and ask, "What's special about this array of numbers?" (On one occasion a girl in my class had a charm bracelet with a tiny aluminium Lo-shu.) I'm simply after establishing that every row, column and diagonal sum to the same line total. For the Lo-shu, this is 15. I next place six puzzles on the whiteboard (Fig. 2), and invite my class to try to solve as many of these as they can in ten minutes. They can tackle them in any order they like.

6		
	5	
	3	4

		10
	7	
4		5

		2
		17
	1	11

14	3	
		13
8	15	

11	1	
9		7
	15	5

16		8
	10	18

Fig. 2

Readers of *Mathematics in School* will have no difficulty in solving these puzzles. Answers from my classes arrive sufficiently slowly to allow me to write each as it is found on the whiteboard along with the name of the solver. The ten minutes is usually up before a class has cracked them all.

I originally intended that the six puzzles be in increasing order of difficulty and had wondered if my pupils would tackle the later ones by means of algebra and solving equations. However, trial and improvement was the popular technique. I deliberately leave the Lo-shu visible from the start of the lesson. Usually, someone spots a transformation that maps the Lo-shu directly onto one of the six. For example, the last of the six puzzles is the Lo-shu doubled and reflected in the rising diagonal. I'll say more about this as a general solution strategy in the second article in the series.

The focus of the lesson comes next. This is to provide a mathematical toolbox that will make it easy for pupils to solve such puzzles. There are five tools in the kit. I shall describe two in this article, and three next time. They all have short proofs, the cleverness of which is often appreciated and commented upon by one or two pupils. I shall present the five tools in order of ease with which my classes seem to grasp them. All come into their own, given the appropriate situation.

Tool 1: The Line Totalizer

The line total of a 3×3 magic square is always three times the number in the central cell.

Proof

Let T be the line total and the numbers in the cells be represented algebraically (Fig. 3).

a	*b*	*c*
d	*e*	*f*
g	*h*	*i*

Fig. 3

Observe that

$$\begin{array}{rrcl} & a + e + i & = & T \\ & b + e + h & = & T \\ ADD & c + e + g & = & T \\ \hline & T + 3e + T & = & 3T \\ & 3e & = & T \end{array}$$

It is instructive to encourage pupils to look back at the solutions to the six puzzles and verify that the line totals are, as claimed, three times the value in the central cell.

At this juncture it's worth looking at the Lo-shu with arithmetic progressions in mind. The key observation sought is that any line through the central cell is in arithmetic progression. I prepare my class for the next proof by taking each of the lines through the central cell of the Lo-shu and writing them out in a manner that emphasizes the essential nature of a three-term arithmetic progression.

1, 5, 9 = (5 – 4), 5, (5 + 4)
2, 5, 8 = (5 – 3), 5, (5 + 3)
3, 5, 7 = (5 – 2), 5, (5 + 2)
4, 5, 6 = (5 – 1), 5, (5 + 1)

Tool 2: The Sequencer

Any line through the central cell is in arithmetic progression.

Proof

Consider the diagonal containing vowels (Fig. 4).

a		
	e	
		i

Fig. 4

Note that a can be related to e by the simple equation $a = e - x$, where x is an integer.
Now $a + e + i = T$, the line total, from which is obtained $i = T - a - e$.
But from the toolbox comes the fact that $3e = T$.
So, $i = 3e - (e - x) - e$.
$\therefore\ i = e + x$, with a little care being taken over the subtraction of the negative number.
The idea behind the previous number work was to increase an appreciation that

$$a, e, i, = (e - x), e, (e + x)$$

is, as claimed, in arithmetic progression.

Repeating this technique on the other diagonal results in Figure 5.

$e - x$		$e - y$
	e	
$e + y$		$e + x$

Fig. 5

Keeping in mind that the lines *not* through the centre also must sum to the line total, $3e$, the remaining cells can now be filled in briskly (Fig. 6).

$e - x$	$e + (x + y)$	$e - y$
$e + (x - y)$	e	$e - (x - y)$
$e + y$	$e - (x + y)$	$e + x$

Fig. 6

To complete the proof, we simply observe that the horizontal and vertical lines through the central cell are, as claimed, in arithmetic progression. The bracketing used has been chosen to make this apparent.

All of the original six puzzles are solved easily using the two tools developed so far. Rather than dwell on them, however, pupils will be keen to try a fresh set (Fig. 7). These are clearly harder than those given previously but armed with the two new tools, all are approachable. One useful feature of a three-term arithmetical progression, which I let my pupils discover for themselves, is that the middle term is the arithmetical mean (the average) of the other two.

17	37	
	21	

	17	23
14		

27	17	
	25	

13		
		19
		9

43		31
		7

44		24
	29	

Fig. 7

I'll conclude this article with three examples for which the two-tool toolbox is inadequate (Fig. 8).

	31	17
	59	

10	15	
		17

	41	15
		35

Fig. 8

I let my classes take these away as 'challenge' problems. Next time I'll complete the toolbox such that all 3 × 3 magic square puzzles can be resolved competently.

Reference

Rose, M. 2009 'Resource Notes 29, Sequential Magic Squares', *Mathematics in School*, **38**, 2, p. 13.

Keywords: Magic squares; Arithmetic progressions; Proof.

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