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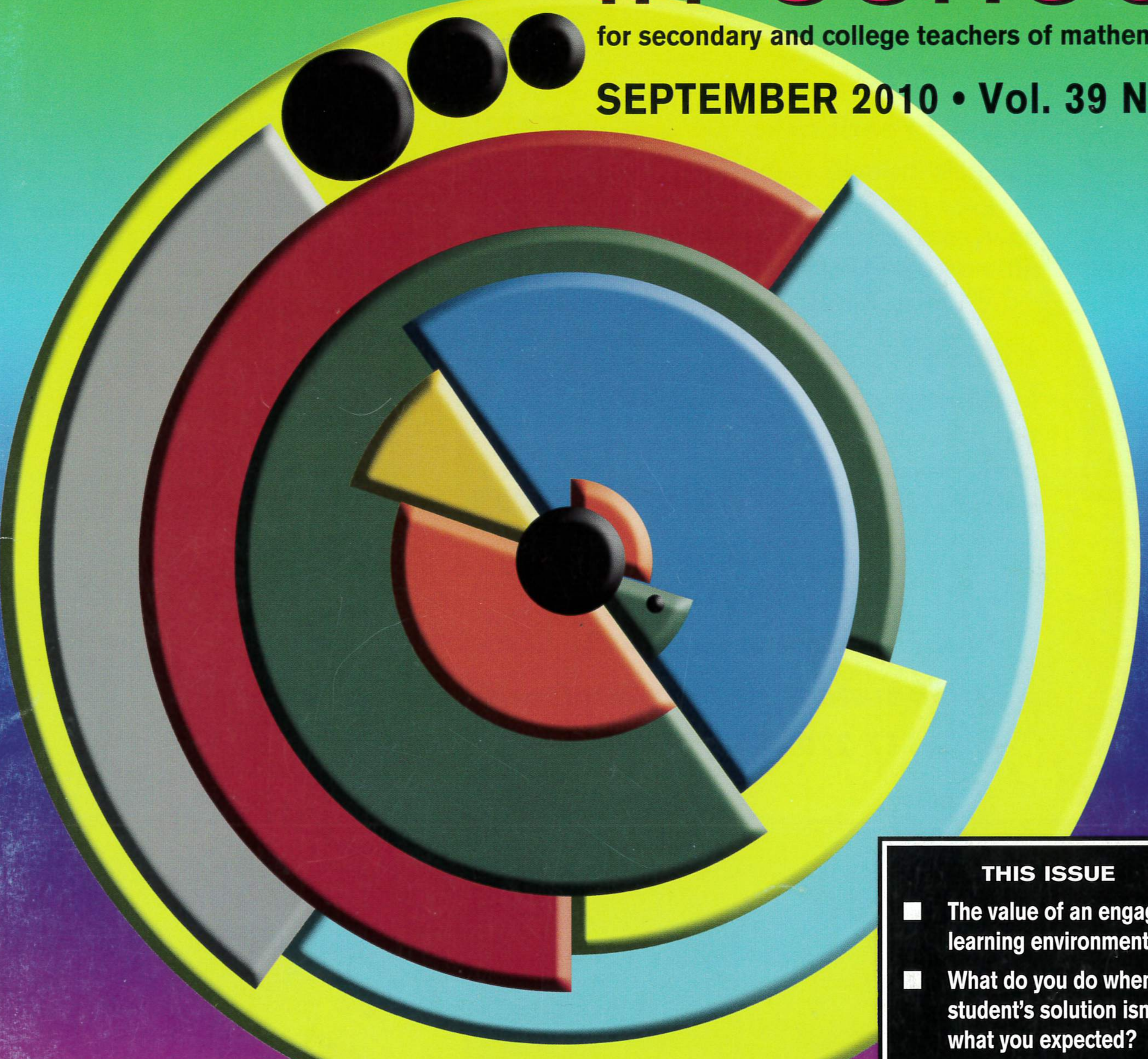

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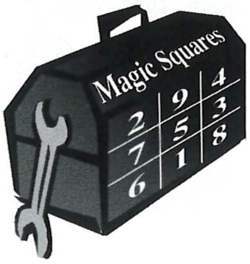

Magic Squares
from a Teaching Point of View – Part 2

by Martin Hansen

Previously, I presented the first two of five tools of a toolbox that make the solving of any 3 × 3 magic square puzzle straightforward.

Tool 1: The Totalizer

The line total of a 3 × 3 magic square is always three times the number in the central cell.

Tool 2: The Sequencer

Any line through the central cell is in arithmetic progression.

I gave mathematical proofs that I use in my teaching of this topic and which explained why these two tools work. I also gave two sets of six puzzles on which to try them out. I concluded with three 'challenge' problems for which the two tools developed last time are inadequate. These I present afresh below (Fig. 8) as they provide the motivation for my second 'fun' lesson on the topic of magic squares.

	31	17
	59	

10	15	
		17

	41	15
		35

Fig. 8 (recap)

My pupils get an initial buzz from realizing that some progress can be made with the left-hand challenge. Two approaches are apparent. Either calculate 3 × 31 to get the line total for the square (Tool 1), or complete the two arithmetic progressions through the centre (Tool 2). Both avenues of attack lead to Figure 9, but then the true difficulty of the puzzle becomes apparent. The middle and the right-hand challenges do not offer any initial easy ground.

	3	
45	31	17
	59	

Fig. 9

Of course, the three will succumb to a determined attack based on the strategies 'trial and improvement' or 'algebra and equation solving'. In fact, having got to Figure 9, there is a convoluted solution that one of my pupils found, which I had not intended. Take the Lo-shu, shown left in Figure 10. Reflect it in the central horizontal line to get the magic square shown centrally in Figure 10. Transform each term of this using $f(x) = 7x - 4$. The solution, shown in Figure 10, results.

2	9	4
7	5	3
6	1	8

6	1	8
7	5	3
2	9	4

38	3	52
45	31	17
10	59	24

Fig. 10

Note that the other two 'challenge' puzzles (going back to Figure 8) will provide counterexamples to the popular misconception that all 3 × 3 magic squares can be obtained by a reflection or rotation of the Lo-shu followed by a transformation of the form $f(x) = mx + c$, for some constants m and c. So tackling them by this difficult method will be in vain. As a method it does not generalize and so is fundamentally flawed. I'll look in detail at the true structure of the space of all 3 × 3 magic squares in the third article in this series.

The third and fourth tools in the toolbox both arise from looking a little harder at the diagram produced when proving the second tool. I've reproduced it as Figure 11, but have shaded out all but three cells. Focus on those three cells.

$e - x$	$e + (x + y)$	$e - y$
$e + (x - y)$	e	$e - (x - y)$
$e + y$	$e - (x + y)$	$e + x$

Fig. 11

Tool 3: The Averager

The mean (average) of any two cells on a broken diagonal is the same as the value in the corner cell not touching that broken diagonal.

Proof

For the broken diagonal shown unshaded in Figure 11,

$$\text{mean} = \frac{\{e+(x+y)\}+\{e+(x-y)\}}{2}$$

$$= \frac{2e+2x}{2}$$

$$= e + x.$$

Now note that $e + x$ is the value in the corner cell that does not touch the broken diagonal. To complete the proof, algebraically work through the other three cases (or invoke a symmetry argument).

Look at the left-hand challenge puzzle from Figure 8. The third tool allows us to discover that the value of the upper left cell is 38 (the mean of 59 and 17). The arithmetic progression on the falling diagonal (Tool 2) yields 24 as the value in the lower right cell. The puzzle can now be resolved easily.

Tool 4: The Centralizer

The corner cell that does not touch a broken diagonal is the middle term of a three-term arithmetic progression. The cells of the broken diagonal are the other two terms.

Proof

Consider the three cells shown unshaded in Figure 11. They are $e + (x - y)$, $e + x$, $e + (x + y)$.

They can be rebracketed as $(e + x) - y$, $e + x$, $(e + x) + y$, which is the form of a three-term arithmetic progression. The middle term, as claimed, is the corner cell that does not touch those of the broken diagonal. To complete the proof, algebraically work through the other three cases (or invoke a symmetry argument).

Look at the middle 'challenge' puzzle (Fig. 8). The third tool states that the mean of 15 and 17, which is 16, is the value of the cell lower left (Fig. 12). Focus on the cell marked z.

10	15	
		17
16	z	

Fig. 12

The fourth tool states that z, 10 and 17 form an arithmetic progression with 10 as middle term. Thus the missing term has to be 3. As the vertical line through the centre is in arithmetic progression (by Tool 2) the central cell must contain the value 9. The remaining values are now easily found and so the middle challenge resolved (Fig. 13).

10	15	2
1	9	17
16	3	8

Fig. 13

Tool 5: The Dot Spotter

Place a star in any cell. The sum of the other two terms in the starred row equals the sum of the other two terms in the starred column. Each is also equal to the sum of the other two terms in any starred diagonal(s).

Proof

Consider the three situations shown in Figure 14.

No Diagonal

a	⋆	c
	e	
	h	

One Diagonal

⋆	b	c
d	e	
g		i

Two Diagonals

a	b	c
d	⋆	f
g	h	i

Fig. 14

Let T be the line total of each row, column and diagonal. In the case 'No Diagonal':

$$a + \star + c = \star + e + h = T$$
$$a + c = e + h.$$

In the case 'One Diagonal':

$$\star + b + c = \star + d + g = \star + e + i$$
$$b + c = d + g = e + i.$$

In the case 'Two Diagonals':

$$a + \star + i = b + \star + h = c + \star + g = f + \star + d$$
$$a + i = b + h = c + g = f + d.$$

To complete the proof, work through the algebra of the other three 'No Diagonal' cases, and the other three 'One Diagonal' cases, or invoke a symmetry argument.

The toolbox is now complete. As with the popular Sudoku puzzles, flipping between the tools or 'solution techniques' is a skill that comes with practice. Indeed, it's the variety of possible lines of attack that makes the puzzles interesting and promotes good discussions when one pupil compares his method with another. Figure 15 provides another set of six puzzles upon which to practise using the tools.

	1	
17	11	

1		
	11	13

		33
	28	26

40		
		24
20		

32		8
	6	

17	2	23

Fig. 15

Note that the last puzzle (Fig. 15) exposes another popular misconception that a line of three can not uniquely define a 3 × 3 magic square. In this position, it does.

The puzzles avoid using zero or negative numbers. In any given puzzle, no number occurs more than once. These are two widely accepted but unwritten rules.

One of the difficulties in providing sets of puzzles is in catering for the wide range of abilities and situations in which they may be used. In my next and final article, I'll provide a deeper insight into the space containing all possible 3 × 3 magic squares that respect the 'unwritten rules', so that readers can construct further puzzles tailored to their audience, quickly and efficiently.

Keywords: Magic squares; Arithmetic progressions; Proof.

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