# mathematics in school

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# Magic Squares

## from a Teaching Point of View – Part 3

by Martin Hansen

In the first two articles in this series, I presented a mathematical toolbox. Its five tools enabled any 3 by 3 magic square puzzle to be solved, provided there were sufficient clues to define a unique solution. To recap, here's the toolbox and six puzzles on which to try it out (Fig. 16).

### Tool 1: The Totalizer

The line total of a $3 \times 3$ magic square is always three times the number in the central cell.

### Tool 2: The Sequencer

Any line through the central cell is in arithmetic progression.

### Tool 3: The Averager

The mean (average) of any two cells on a broken diagonal is the same as the value in the corner cell not touching that broken diagonal.

### Tool 4: The Centralizer

The corner cell not touching a broken diagonal is the middle term of a 3-term arithmetic progression. The cells of the broken diagonal are the other two terms.

### Tool 5: The Dot Spotter

Place a star in any cell. The sum of the other two terms in the starred row equals the sum of the other two terms in the starred column. Each is also equal to the sum of the other two terms in any starred diagonal(s).

| | | |
|---|---|---|
| | | 7 |
| | 23 | 33 |
| | | |

| | | |
|---|---|---|
| 33 | | |
| | | |
| 23 | | 7 |

| | | |
|---|---|---|
| 23 | 7 | 33 |
| | | |
| | | |

| | | |
|---|---|---|
| | | |
| 33 | | |
| 7 | | 23 |

| | | |
|---|---|---|
| | | |
| 33 | | 7 |
| | | 23 |

| | | |
|---|---|---|
| | 23 | |
| | | 7 |
| | | 33 |

**Fig. 16**

The puzzles of Fig. 16 are challenging, especially the one lower left for which 'The Centralizer' is the tool needed to get going. For classroom use and homework exercises my articles have provided an insufficient number of puzzles, and certainly not enough of an easier nature. In this final instalment I will reveal an ingenious method that will empower readers to create their own. It will also provide a compact way of providing answers.

It is well known that the set of all possible $3 \times 3$ magic squares with integer entries form a mathematical construct known as a vector space. However, this space includes magic squares with negative and duplicate numbers. I rather dislike changing the rules of the puzzle to suit the mathematics. Duplicates, negatives, zero and fractions are all taboo. One of the skills of a mathematician should be an ability to fit the mathematics to the task at hand.

In three steps, I will now explain how to generate magic squares with the desired properties. In practice it's quick and easy to apply. It also gives more insight into the underlying structure of these fascinating objects.

**Step 1**

Decide what number the central cell is to contain, for example 19. To be consistent with the notation of the previous articles, this is the $e$ number.

**Step 2**

Decide upon two further numbers, $x$ and $y$, such that $e > x > y$ and $y \neq \frac{1}{2}x$.

For example, with $e = 19$, I'll pick $x = 8$ and $y = 2$. Again, the algebraic letters used are consistent with that of my previous articles where the common difference of the arithmetic progression on the falling diagonal was $x$ and on the rising diagonal, $y$. The three parameters, $e$, $x$, and $y$ specify a unique magic square. In many ways this is like specifying a point in three dimensions, so I'll write it as $(e, x, y)$ or, for the example, (19, 8, 2).

**Step 3**

With reference to Figure 17, position $e$, $x$, $y$, $(x - y)$ and $(x + y)$ as shown. With the example numbers, they are 19, 8, 2, 6 and 10, respectively.

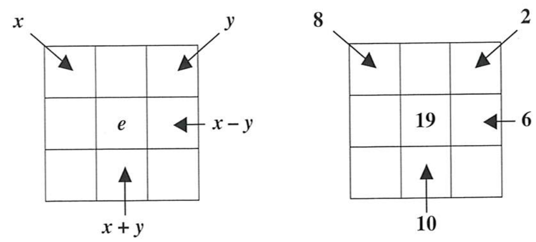


**Fig. 17**

Complete the four arithmetic progressions, as indicated by the arrows in Figure 17.

Each has central term, $e$, and common differences are $x$, $y$, $(x - y)$ and $(x + y)$.

The result, (19, 8, 2) is shown below, in Figure 18.

(19, 8, 2)

| | | |
|---|---|---|
| 11 | 29 | 17 |
| 25 | 19 | 13 |
| 21 | 9 | 27 |

(5, 3, 1)

| | | |
|---|---|---|
| 2 | 9 | 4 |
| 7 | 5 | 3 |
| 6 | 1 | 8 |

**Fig. 18**

The magic square generated by this method is always in the 'standard position' as described by (5, 3, 1) which is the Lo-shu, shown above (Fig. 18). In 'standard position', the smallest number is at the centre of the lower row and the second smallest is found top row left. Note that the location of the third smallest is not necessarily found middle row right. Generate (6, 3, 2) to see this.

As puzzle composer, reflect and rotate the 'standard position' creation, if desired. Blank out a few numbers. For the puzzle to have a unique solution all but three numbers can be blanked out but those three must satisfy two criteria. Firstly, they must not all lie in a line through the centre cell. Nor may they lie on a broken diagonal *and* the corner cell that does not touch that broken diagonal. Before discussing the latter point you may wish to generate a few puzzles of your own.

| | | |
|---|---|---|
| | 29 | |
| | | 13 |
| 21 | | |

**Fig. 19**

Suppose that (19, 8, 2) is to be used to generate a puzzle but that, in error, had all but the three numbers 13, 21 and 29 removed (Fig. 19). Why is this violating magic square puzzle etiquette?

Obviously (19, 8, 2) is a solution, but so is its reflection in the rising diagonal. In itself, it could be argued that this is not a significant problem. However, more seriously, Figure 20 shows two further unintended solutions (15, 8, 6) and (23, 8, 2)*. The asterisk indicates that it's a reflection or rotation of (28, 8, 2). That is, it's (23, 8, 2) but not in 'standard position'.

(15, 8, 6)

| | | |
|---|---|---|
| 7 | 29 | 9 |
| 17 | 15 | 13 |
| 21 | 1 | 23 |

(23, 8, 2)*

| | | |
|---|---|---|
| 15 | 29 | 25 |
| 33 | 23 | 13 |
| 21 | 17 | 31 |

**Fig. 20**

Figure 20 provides only two examples of further solutions. There are others. If you find this topic as interesting as I do you may like to try to track them down. However, from a teaching point of view, I feel that the topic has been taken as far as required in order to introduce magic squares in a mathematical manner to pupils.

To finish, here are the answers to the six puzzles of Fig 16.

(23, 16, 6)*, (20, 13, 3)*, (21, 12, 2)*, (17, 10, 6)*, (20, 10, 3)* and (25, 10, 8)*.

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