Lesson 3

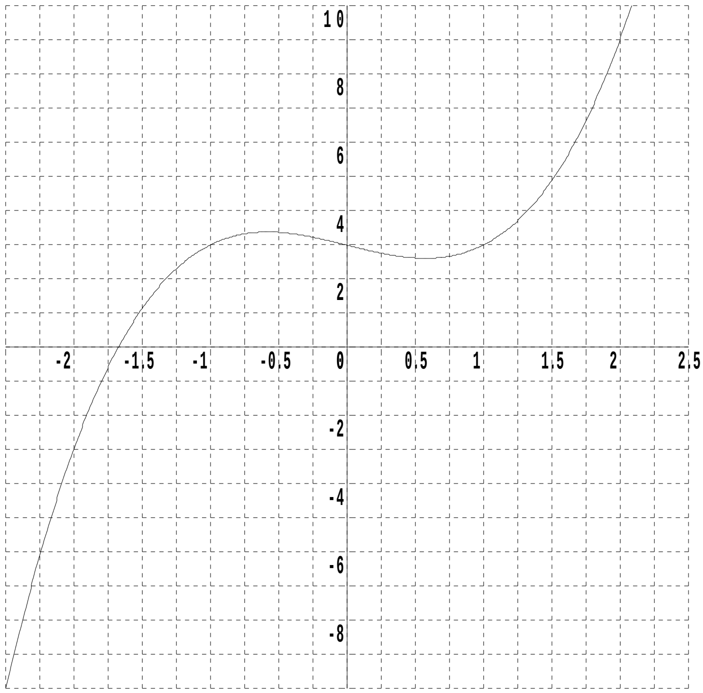
3.1 Numerical Methods Exercise

Question 1

Consider the function

$$f(x) = x^3 - x + 3$$

The graph of this function suggests that it has a root in $-2 \leqslant x \leqslant -1$



(i) Calculate **(a)** $f(-2)$

(b) $f(-1)$

(ii) How do the part **(i)** answers prove that there is a root in $-2 \leqslant x \leqslant -1$?

(iii) Prove that a root is in the interval $-1.8 \leqslant x \leqslant -1.6$.

Question 2

This question is about the function

$$g(x) = \ln x - \frac{1}{x}$$

(i) On one graph sketch the curves

(a) $y = \ln x$

(b) $y = \frac{1}{x}$

to show that $g(x) = 0$ has a root.

(ii) Calculate **(a)** $g(1.7)$

(b) $g(1.8)$

(iii) How do the part **(ii)** answers prove that there is a root in $1.7 \leqslant x \leqslant 1.8$?

(iv) Show that $g(x) = 0$, implies

$$x = e^{\frac{1}{x}}$$

(v) Using the iterative formula

$$x_{n+1} = e^{\frac{1}{x_n}}$$

$$x_0 = 1.4$$

find

(a) $x_1 =$

(b) $x_2 =$

(c) $x_3 =$

(vi) Keep iterating to find a root of $g(x) = 0$ to about eight decimal places.

Question 3
This question is about the function

$$h(x) = x^2 - \sin x$$

where x is measured in **RADIANS**.

(i) On one graph sketch the curves

(a) $y = x^2$

(b) $y = \sin x$

to show that $h(x) = 0$ has a root.

(ii) Use the iteration formula

$$x_{n+1} = \frac{\sin x_n}{x_n}$$

$$x_0 = 1$$

to calculate

$x_1 =$
$x_2 =$
$x_3 =$
$x_4 =$
$x_5 =$

(iii) To prove that a root is 0.8767 to four decimal places, calculate;

(a) $h(0.87665) =$

(b) $h(0.87675) =$

and observe that as one answer is negative and the other positive there is at least one root in the interval $0.87665 \leqslant x < 0.87675$ and that as all numbers in this interval round to 0.8767 the root is indeed 0.8767, correct to four decimal places.

This proof is a technique to remember, including the worded reasoning.

Question 4
Initially the number of fish in a lake is 500 000.

The population is then modelled by the recurrence relation;

$$u_{n+1} = 1.05\,u_n - d$$
$$u_0 = 500\,000$$

In this relation u_n is the number of fish in the lake after n years and d is the number of fish which are caught each year.

Given that $d = 15\,000$,

(a) calculate u_1, u_2 and u_3 and comment briefly on your results.

Given instead that $d = 100\,000$,

(b) Show that the population of fish dies out during the sixth year.

(c) find the value of d which would leave the population each year unchanged.

Question 4 is an Edexcel examination question from January 2002.

Question 5

(i) Use the iteration formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right)$$

$$with\ x_1 = 2$$

to calculate

$x_2 =$

$x_3 =$

$x_4 =$

$x_5 =$

$x_6 =$

(ii) It is thought that this iterative sequence may converges to $\sqrt{n}$ where *n* is an integer.

If this is so, what is *n* ?

(iii) At the fixed point, *X*,

$$x_{n+1} = x_n = X$$

By replacing x_{n+1} and x_n with *X* in the iterative formula, rearrange the formula and so prove that that it does indeed converge to your part **(ii)** answer.

Question 6

Let

$$f(\theta) = \sin\theta + 1 - \theta$$

where θ is in **<u>RADIANS</u>**.

(i) Calculate $f(1)$ and $f(2)$ and explain how this proves that

$$f(\theta) = 0$$

has a solution between $\theta = 1$ and $\theta = 2$.

(ii) Use the iterative formula

$$\theta_{n+1} = \sin\theta_n + 1$$

to find this root correct to 5 decimal places.

(iii) Prove that your part **(ii)** answer is correct using the ***Question 3, part (iii)*** technique, ***including some worded reasoning***.