**Lesson 5**



## 5.1 Iteration Theory

In general, iterations are of the form;

$$x_{n+1} = f(x_n)$$

Once sufficiently near the fixed point of an iteration, whether the iteration will convergence or diverge depends upon the gradient of

$$y = f(x)$$

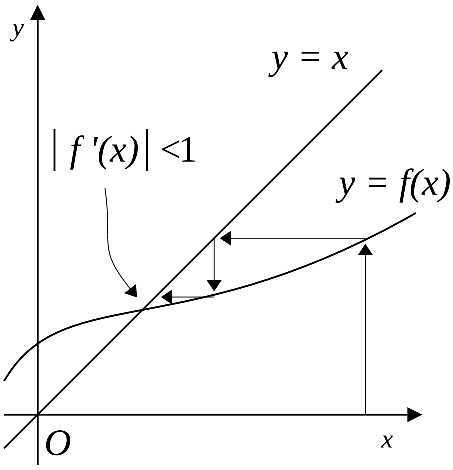
at the intersection with

$$y = x$$

The condition for convergence is that at that point of intersection,

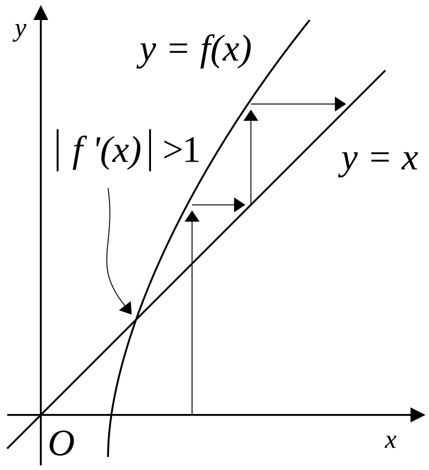
$$|f'(x)| < 1$$

Diagrams of iterations will be investigated shortly - look out for these situations;



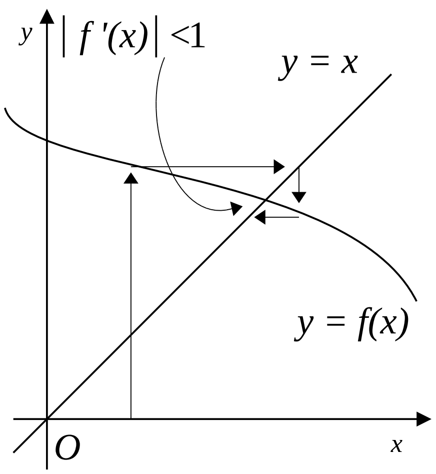

$|f'(x)| < 1$

*staircase converges*



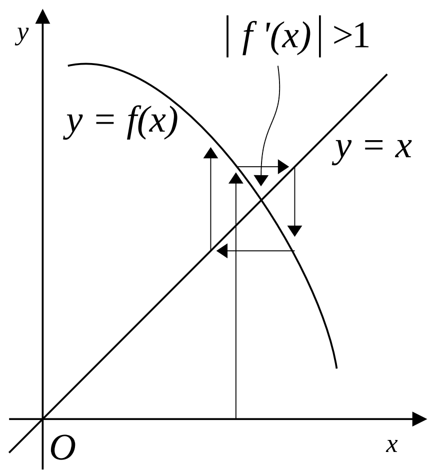

$|f'(x)| > 1$

*staircase diverges*




$|f'(x)| < 1$

*cobweb converges*




$|f'(x)| > 1$

*cobweb diverges*

## 5.2 Picturing Iteration : Cobweb & Staircase Diagrams

**An Investigation**

Consider the iterative relation;

$$x_{n+1} = \frac{x_n^2 - 2x_n + 5}{4}$$

which you are going to iterate from four different starting points.

| $x_0$ | 1 | 3 | 5 | 5.5 |
|---|---|---|---|---|
| $x_1$ | | | | |
| $x_2$ | | | | |
| $x_3$ | | | | |
| $x_4$ | | | | |
| $x_5$ | | | | |
| $x_6$ | | | | |

**( i )** Complete the table, each column being from a different initial value, $x_0$

**( ii )** Given that

$$g(x) = \frac{x_n^2 - 2x_n + 5}{4}$$

differentiate to find $g'(x)$

**( iii )** Find $g'(1)$ and $g'(5)$

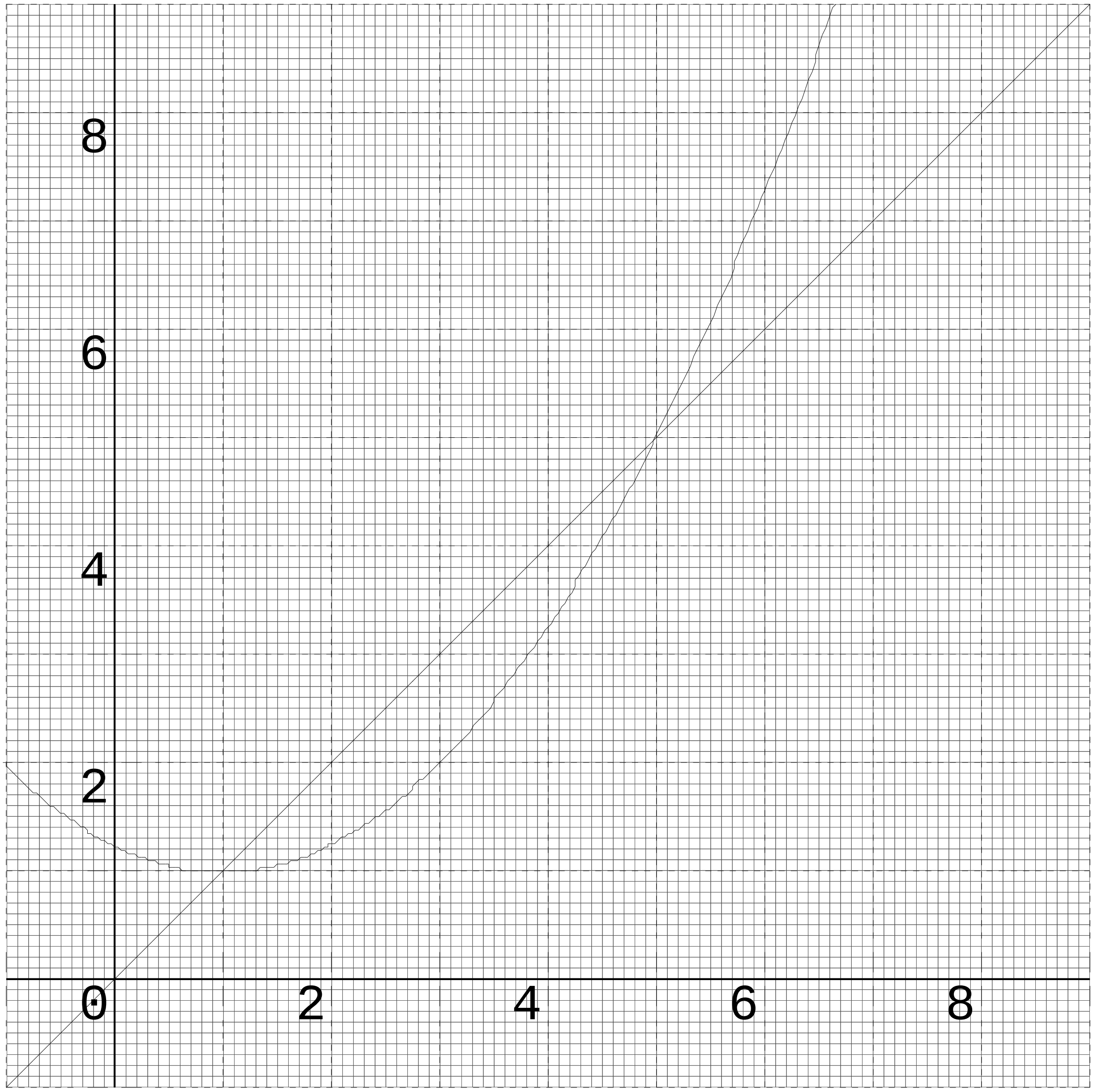
Plotted below is the curve;

$$y = \frac{x^2 - 2x + 5}{4}$$

and the line;

$$y = x$$

**( iv )** Plot your iterations onto this graph.



## 5.3 Exercise

**Question 1**

Consider the iterative relation;

$$x_{n+1} = \frac{12}{x_n} + 1$$

$$x_0 = 6$$

**( a )** Calculate to four decimal places;

| | | | | | |
|---|---|---|---|---|---|
| **( i )** | $x_1 =$ | | **( iv )** | $x_4 =$ | |
| **( ii )** | $x_2 =$ | | **( vi )** | $x_5 =$ | |
| **( iii )** | $x_3 =$ | | **( vii )** | $x_6 =$ | |

**( b )** Plot a cobweb diagram of the iterative process for $x_0$, $x_1$, $x_2$, ... , $x_6$. Use the graph on the following page.

**( c )** Prove that 4 is a fixed point of the iteration by showing that if $x_n = 4$ ***exactly***, then $x_{n+1} = 4$ ***exactly***.

**( d )** At a fixed point, *X*, of the iteration

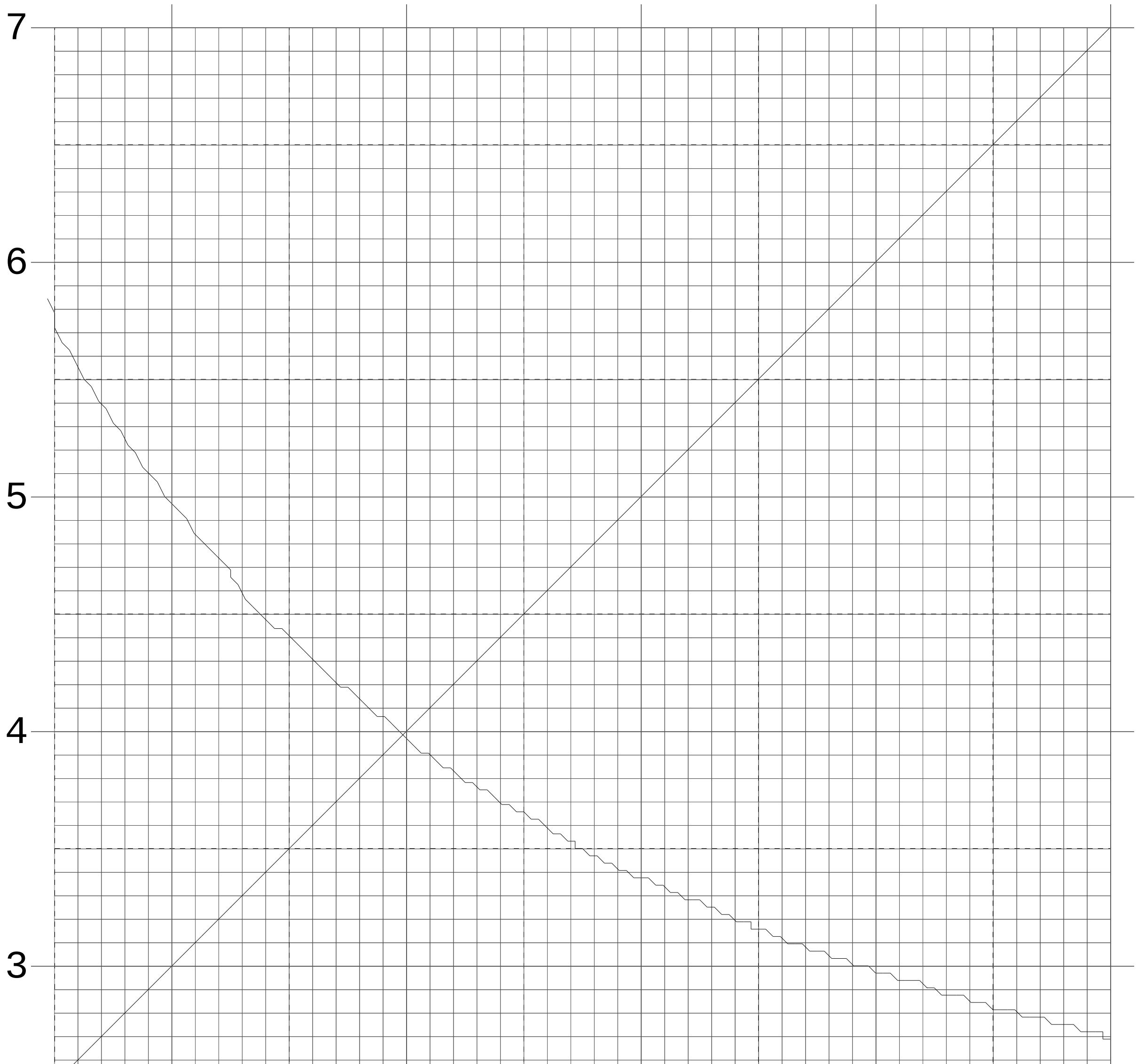
$$x_{n+1} = x_n = X$$

By replacing $x_{n+1}$ and $x_n$ with *X* in the iterative formula, rearrange the formula and so prove that, in addition to the fixed point at 4 there is another fixed point of this iteration. State the value of this other fixed point.

**( e )** With

$$f(x) = \frac{12}{x} + 1$$

calculate the value of the gradient at each of the two fixed points. Explain what this tells you about each fixed point.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**Question 2**

Consider the iterative relation;

$$x_{n+1} = ln(100\,x_n)$$

$$x_0 = 0.5$$

**( a )** Calculate to four decimal places;

**( i )** $x_1 =$ **( ii )** $x_2 =$ **( iii )** $x_3 =$

**( b )** Plot a cobweb diagram of the iterative process for $x_0$, $x_1$, $x_2$, $x_3$.
Use the graph on the following page.

**( c )** Given that

$$f(x) = ln(100\,x) - x$$

prove that the fixed point of the iteration is, to two decimal places 6.47.

**( d )** Let

$$g(x) = ln(100\,x)$$

Use the rule of logs which states;

$$ln(ab) = ln\,a + ln\,b$$

to help determine $g'(x)$ and hence show that the fixed point is an attractor.

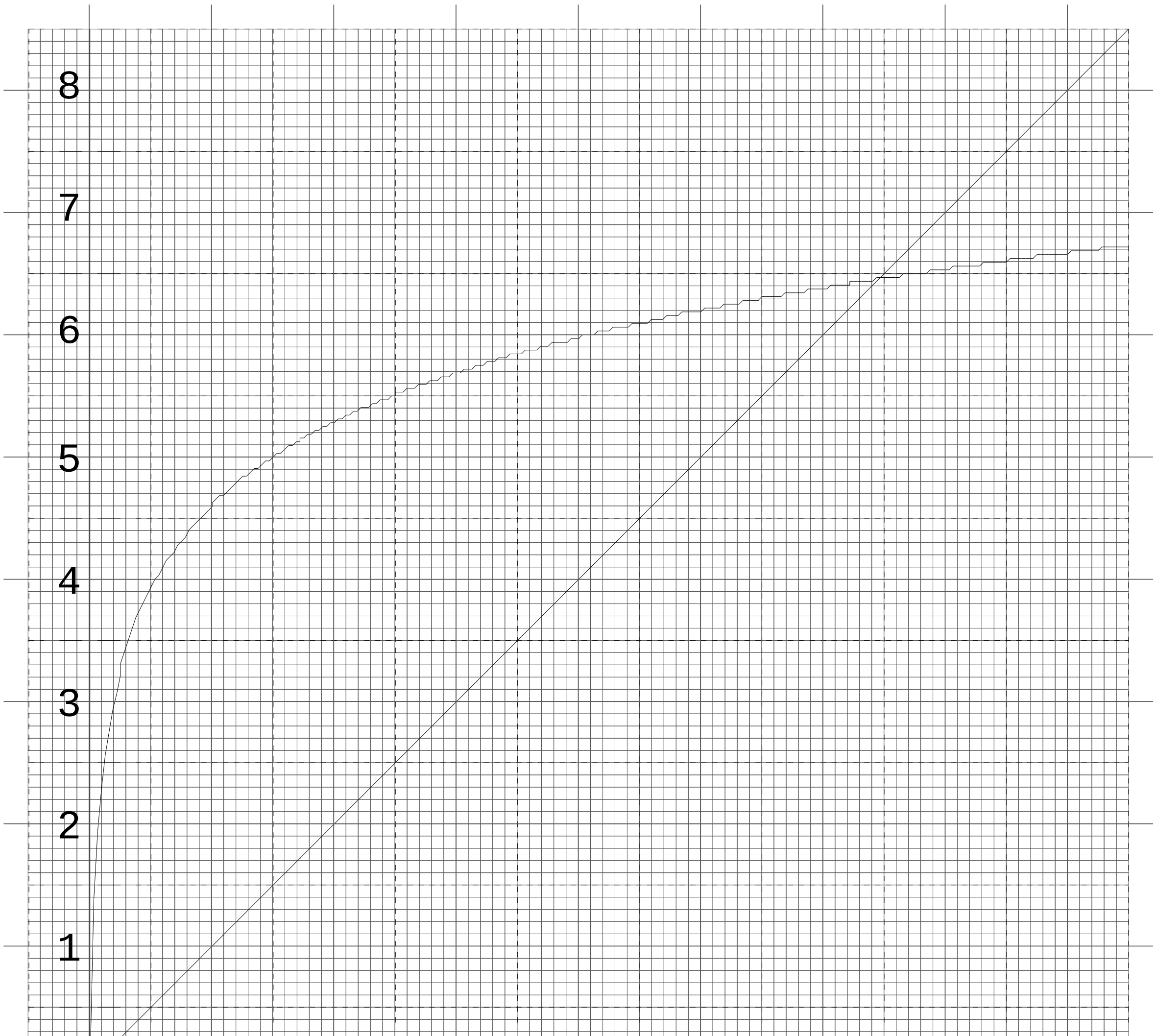
**( e )** Predict what will happen if the iteration starts with

**( i )** $x_0 = 8$

**( ii )** $x_0 = 0.05$

**( iii )** $x_0 = 0.01$

Test your predictions and comment.



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**Question 3**

**( a )** Show that

$$f(x) = x \ln x - 3$$

has a root between 2 and 3.

**( b )** Show that

$$x \ln x - 3 = 0$$

can be rearranged to give the iterative relationship

$$x_{n+1} = \frac{3}{\ln x_n}$$

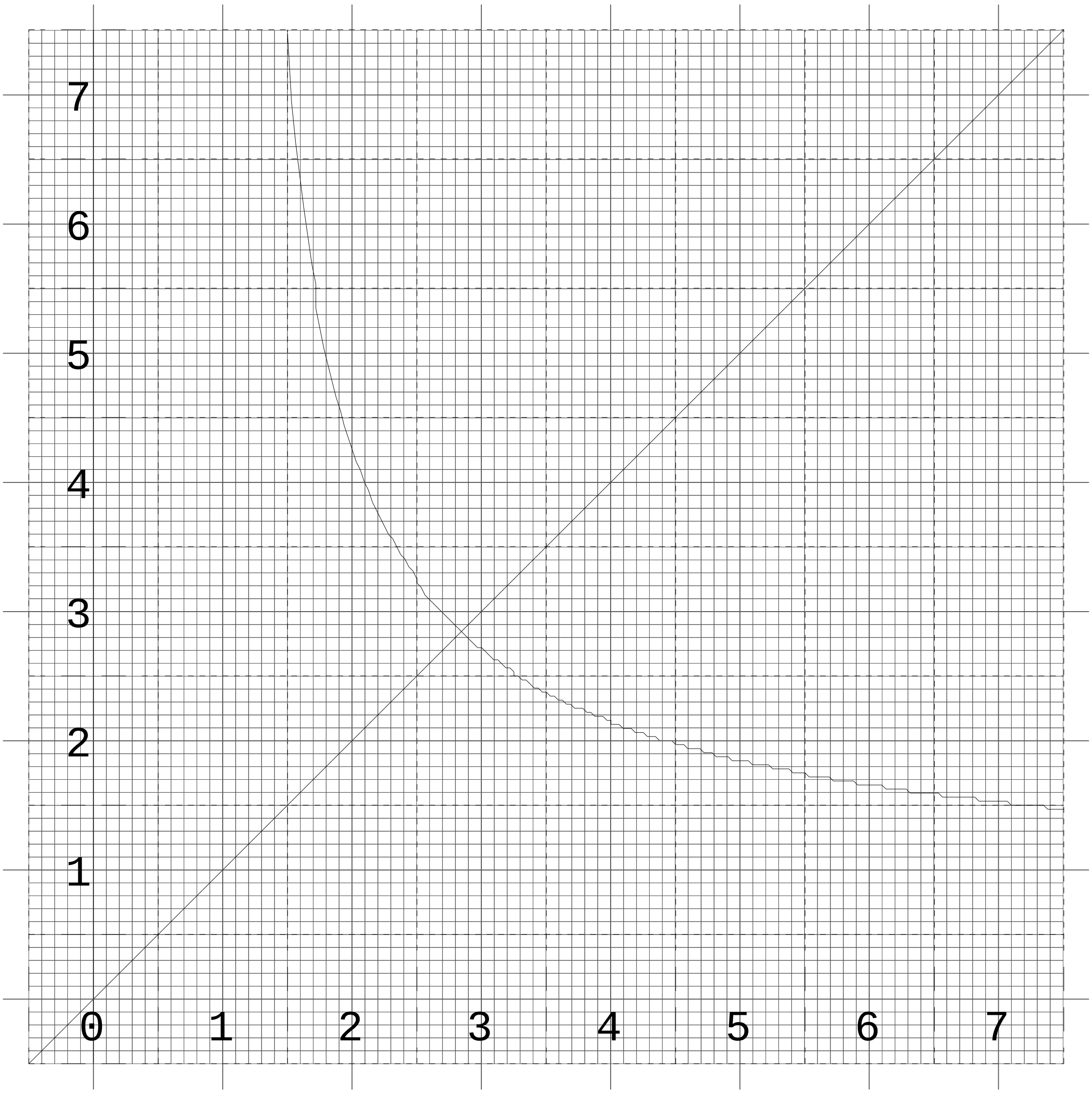
**( c )** With $x_0 = 2$, calculate $x_1$ and $x_2$.

**( d )** Plot the iterative process $x_0$, $x_1$ and $x_2$ on a cobweb diagram. Use the graph on the following page.

**( e )** What you have done so far, indicates that this iteration may be problematical. Explain what the nature of the problem is.

**( f )** Iterate sufficiently to locate the root of $f(x)$ correct to 3 decimal places. Write down the your answer.

**( g )** Prove that your **part ( f )** answer is indeed the root correct to 3 decimal places.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