Lesson 5

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics In Algebra

5.1 Quadratic Equation Roots

A quadratic equation is of the form; $y = ax^2 + bx + c$

In this equation x and y are variables, whereas a, b and c are constants.
The graph of such an equation is a ***quadratic curve***, also called a ***parabola***.
It's a useful shape, used in, for example, car headlights, electric heaters, radio telescopes and solar furnesses.
Often the mathematical interested is where (if anywhere) a given quadratic curve crosses the x-axis. It does this when it has zero height; that is, when y is zero. When we talk about ***solving a quadratic equation*** we mean finding the values of x (often called the *roots*) for which $y = 0$.

Quadratic equations, when written in the form, $ax^2 + bx + c = 0$ can be solved by

(i) Factorisation into two pairs of brackets,
(ii) Completing the square.
(iii) Using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5.2 Example

Find the roots of the equation $x^2 + 14x + 40 = 0$ in each of the following ways;

(i) By factorisation into two pairs of brackets,

$$x^2 + 14x + 40 = 0$$

[2 marks]

(ii) By completing the square,

$$x^2 + 14x + 40 = 0$$

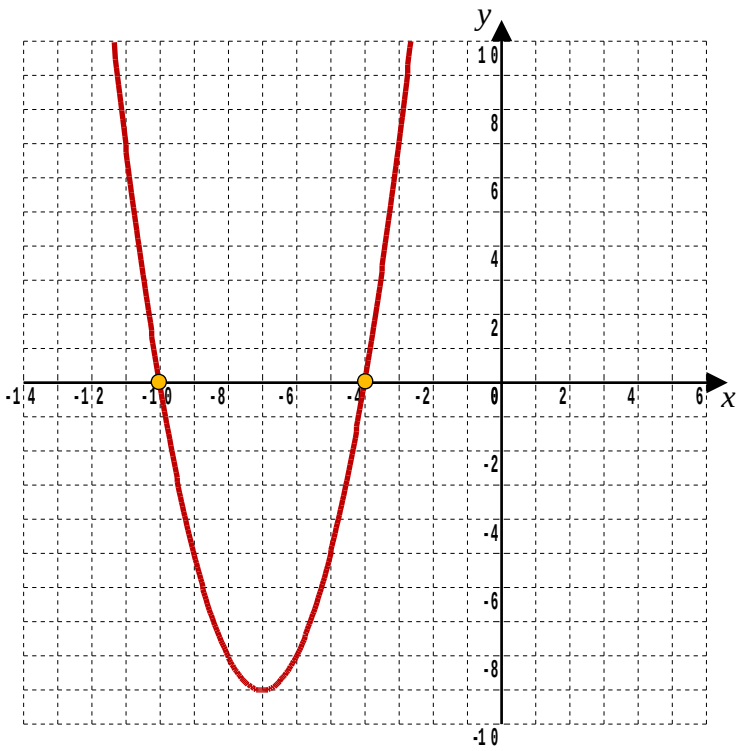
[2 marks]

(iii) Using the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x^2 + 14x + 40 = 0$$

[2 marks]

The graph of $y = x^2 + 14x + 40$ is plotted below;



The roots $x = -10$, $x = -4$ correspond to where the curve cuts the x-axis.

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

By first factorising the quadratic, find the solutions to the following equations,

(i) $9x^2 - 16x + 7 = 0$

[3 marks]

(ii) $14x^2 + 11x + 2 = 0$

[3 marks]

Question 2

Use the method of completing the square to find the exact roots of the equation,

$$y = x^2 - 6x + 1$$

(The roots are the values of x that make $y = 0$)

[3 marks]

Question 3

Use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the exact roots of the equation,

$$y = 7x^2 - 3x - 2$$

[3 marks]

Question 4

Simon has decided to use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the equation $x^2 + x + 1 = 0$ and find where the graph of $y = x^2 + x + 1$ crosses the x-axis.

Simon reports that the formula "goes wrong". Explain what he means by this and also what you can thus deduce about where the graph crosses the x-axis.

[4 marks]

Question 5

In general, the part of the formula for solving quadratic equations that sits under the square root symbol is called the discriminant, *D*.

Thus, $D = b^2 - 4ac$

For a particular quadratic equation, *D* can have a positive value, or a negative value or it can be equal to zero.

If *D* has a positive value, that is, $D > 0$, then

- the quadratic equation has two real roots.

If *D* has a value of exactly zero, that is $D = 0$, then

- the quadratic equation has one real root.
  (Annoyingly and confusingly this situation is often referred to as being of one repeated real root or two identical real roots)

If *D* has a negative value, that is, $D < 0$, then

- the quadratic equation has zero real roots.

For each of the following quadratic equations, calculate the value of the discriminant, *D*, and hence state the number of distinct real roots that equation has.

DO NOT SOLVE THE EQUATIONS !

(i) $x^2 + 4x - 5 = 0$

(ii) $x^2 + 4x + 5 = 0$

(iii) $2x^2 + x - 5 = 0$

(iv) $x^2 + 4x + 4 = 0$

(v) $3x^2 + 4x + 2 = 0$

(vi) $4x^2 - 25 = 0$

[6 marks]

Question 6

A-Level Examination Question from June 2009, C1, Q6 (Edexcel)

The equation $x^2 + 3px + p = 0$, where p is a non-zero constant, has equal roots.
Find the value of p.

[4 marks]

Question 7

A-Level Examination Question from January 2018, C12, Q4 (a)

The equation,

$$(p - 2)x^2 + 8x + (p + 4) = 0, \text{ where } p \text{ is a constant}$$

has no real roots.

Show that p satisfies $p^2 + 2p - 24 > 0$

[3 marks]

Question 8

A-Level Examination Question from January 2016, C12, Q13(a) (Edexcel)

The equation $k\left(3x^2 + 8x + 9\right) = 2 - 6x$, where k is a real constant, has no real roots. Show that k satisfies the inequality

$$11k^2 - 30k - 9 > 0$$

[4 marks]

Question 9

A-Level Examination Question from May 2016, C1, Q8 (Edexcel)

The straight line with equation $y = 3x - 7$ does not cross or touch the curve with equation $y = 2px^2 - 6px + 4p$, where p is a constant.

Show that, $4p^2 - 20p + 9 < 0$

[4 marks]

Question 10

The circle, C, with centre $(3, -5)$ and radius 5 has equation,

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

The line with equation $y = kx$, where k is a positive constant, cuts C at a single point.
Find the exact value of k.

[6 marks]