Lesson 6

Additional Mathematics
A-Level Pure Mathematics : Year 1
Topics In Algebra

6.1 The Discriminant

In lesson 5 it was noted that, when solving quadratic equations, the number of roots (solutions) is not always two.
Here is a recap of the theory giving rise to this observation;

The starting point, from GCSE, is to recall that a generalised quadratic;

$$a x^2 + bx + c = 0$$

has solutions given by;

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The piece of formula under the square root sign determines the number of roots.
This crucial piece of formula is termed the discriminant;

$$D = b^2 - 4ac$$

Here are three examples to illustrate the dramatic effect the discriminant has on the number of roots:

$x^2 + x + 1 = 0$	$x^2 + 2x + 1 = 0$	$x^2 + x - 2 = 0$
$a = 1$	$a = 1$	$a = 1$
$b = 1$	$b = 2$	$b = 1$
$c = 1$	$c = 1$	$c = -2$
$x = \frac{-1 \pm \sqrt{-3}}{2}$	$x = \frac{-2 \pm \sqrt{0}}{2}$	$x = \frac{-1 \pm \sqrt{9}}{2}$
$x = \{ \}$	$x = \{-1\}$	$x = \{-2, 1\}$
No Roots	One Root	Two Roots
$D < 0$	$D = 0$	$D > 0$
D is Negative	D is Zero	D is Positive

Notes

- Sometimes *One Root* is referred to as *a repeated root* or *repeated roots.*
- Geometrically the roots are where the graph of the quadratic crosses the *x*-axis.
- The theory is often used the other way around. For example, if told that an equation has a repeated root then it can immediately be deduced that,

$$D = 0$$

That is, $\quad b^2 - 4ac = 0$

6.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 46

Question 1

For each of the following equations,

(a) Calculate the value of the discriminant,

(b) State if the equation has 0, 1 or 2 roots.

DO NOT SOLVE THE EQUATIONS !

(i) $x^2 + 4x - 3 = 0$

(ii) $x^2 - 2x + 8 = 0$

(iii) $x^2 + 6x + 9 = 0$

(iv) $(x + 4)(x + 3) + 1 = 0$

(v) $\dfrac{(x - 2)}{(x + 1)} = x + 3$

(vi) $9x^2 + 30x + 25 = 0$

[6 marks]

Question 2

The following equation has no real roots

$$x^2 - 3x - k = 0$$

where k is some fixed real constant.

Show that k must be less than -2.25

[3 marks]

Question 3

The following equation has two distinct real roots;

$$x^2 + kx + 4 = 0$$

where k is some fixed real constant.

(i) Show that $k^2 > 16$

[2 marks]

(ii) Which of the following equations will have two distinct real roots ?

(a) $x^2 + 5x + 4 = 0$

(b) $x^2 + 3x + 4 = 0$

(c) $x^2 - x + 4 = 0$

(d) $x^2 - 7x + 4 = 0$

[2 marks]

(iii) Solve by completing the square;

$$x^2 + 6x + 4 = 0$$

[2 marks]

Question 4

The following equation has a repeated root;

$$x^2 + (2k + 10)x + \left(k^2 + 5\right) = 0$$

where k is some fixed constant.

Determine the value of k.

[3 marks]

Question 5

A-Level Examination Question from January 2005, C1, Q3 (Edexcel)

Given that the equation

$$kx^2 + 12x + k = 0$$

where k is a positive constant, has equal roots, find the value of k.

[4 marks]

Question 6

A-Level Examination Question from May 2006, C1, Q8 (Edexcel)

The equation

$$x^2 + 2px + (3p + 4) = 0$$

where p is a positive constant, has equal roots.

(a) Find the value of p.

[4 marks]

(b) For this value of p, solve the equation

$$x^2 + 2px + (3p + 4) = 0$$

[2 marks]

Question 7

Explain why

$$(x + 4)^2 + 1$$

is always positive, no matter what the value of x.

[2 marks]

Question 8

By completing the square, show that the expression

$$x^2 + 2x + 5$$

is positive for all real values of x.

[4 marks]

Question 9

$$f(x) = x^2 + (k + 5)x + 4k$$

where k is a real constant.

(i) Show that the discriminant can be expressed in the form

$$(k - a)^2 + b$$

where a and b are positive integers.

[3 marks]

(ii) Explain how your part **(i)** answer reveals that $f(x) = 0$ will always have two real roots.

[2 marks]

Question 10
Without solving the equation, how many roots does the following equation have ?

$$6x^2 + 7x - 5 = 0$$

Justify your answer.

[3 marks]

Question 11
Consider the equation

$$(k + 1)x^2 + kx + k + 1 = 0$$

where k is a constant.

This equation has a repeated root.

Determine the possible values of k.

[4 marks]