2.1 The Rules of Logs

The value of many "nice" logarithms can be determined immediately and without the use of a calculator by making use of the following rule;

The "Jump out of *logs*" Rule

$$\log_c a = b \quad \Leftrightarrow \quad c^b = a$$

Proof

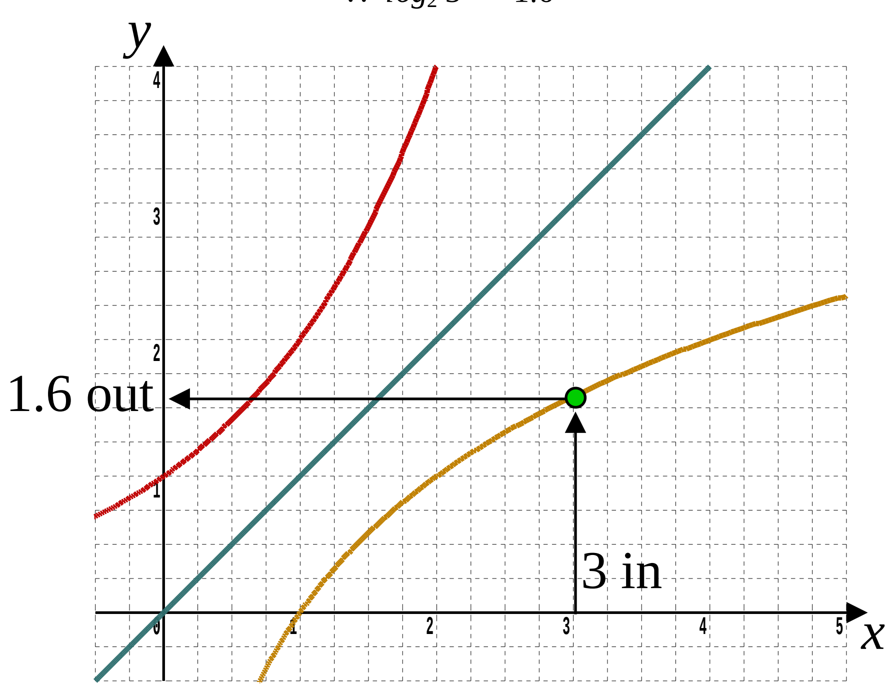
$$\log_c a = b$$
$$\therefore \quad c^{\log_c a} = c^b$$
$$a = c^b \qquad \square$$

For example, to determine the value of $\log_2 32$ the question translates into "2 to what power is 32 ?" which yields the answer 5 because $2^5 = 32$

For more "awkward" logarithms the use of a calculator can still be avoided and a value for the logarithm found by plotting a graph and reading off the required value. For example, to find the value of $\log_2 3$, plot $y = 2^x$ (in red) and then reflect in the line $y = x$ to obtain the graph of the inverse function $y = \log_2 x$.

Then go up from $x = 2$ to the reflection graph and across to 1.6

$$\therefore \log_2 3 = 1.6$$



Whilst drawing a graph aids understanding rather too long winded.
Using a calculator, you can determine a better approximate value of any logarithm. For example, my Casio Classwiz fx-991ex calculator tells me that,

$$log_2 3 = 1.584962501...$$

This is an irrational number, like π and $\sqrt{2}$

With such numbers, if an exact answer is requested, do NOT move into decimals.

The exact value of $log_2 3$ is $log_2 3$ in the same sense that the exact value of π is π.

This in turn necessitates a need to be able to simplify expressions such as,
$log_2 3 + 2\, log_2 5$ without resorting to decimals.
(Decimals can only yield an approximation to the exact answer)

In this case $log_2 3 + 2\, log_2 5 = log_2 75$

To efficiently perform such calculations mathematicians have established a collection rules; “The Rules of Logs”

2.2.1 The First Rule

$$log_c (ab) = log_c a + log_c b$$

Proof : Let $log_c a = x$ and $log_c b = y$ in which case

$$c^x = a \quad \text{and} \quad c^y = b$$

by using the “Jump out of *logs* rule”

By the law of indices,

$$ab = c^x \times c^y \Leftrightarrow ab = c^{x+y}$$

Now, using the “Jump out of *logs* Rule” backwards,

$$log_c (ab) = x + y$$

Giving, $$log_c (ab) = log_c a + log_c b \qquad \square$$

Illustration: Verify that; $log_2 4 + log_2 16 = log_2 64$ where $4 \times 16 = 64$

2.2.2 The Second Rule

$$log_c\left(\frac{a}{b}\right) = log_c\, a - log_c\, b$$

Proof : Let $log_c\, a = x$ and $log_c\, b = y$ in which case

$c^x = a$ and $c^y = b$

by using the "Jump out of *logs* rule"

By the laws of indices,

$$\frac{a}{b} = \frac{c^x}{c^y} \Leftrightarrow \frac{a}{b} = c^{x-y}$$

Now, using the "Jump out of *logs* Rule" backwards,

$$log_c\left(\frac{a}{b}\right) = x - y$$

Giving, $log_c\left(\frac{a}{b}\right) = log_c\, a - log_c\, b$ □

Illustration: Verify that, $log_3 81 - log_3 27 = log_3 3$ where $\frac{81}{27} = 3$

2.2.3 The Third Rule

$$log_c\, a^n = n\, log_c\, a$$

Proof : Let $log_c\, a = x$ in which case $c^x = a$ ("Jump out of *logs* rule")

By the laws of indices,

$$a^n = (c^x)^n \Leftrightarrow a^n = c^{nx}$$

And so, $log_c\, a^n = n\, x$ ("Jump out of logs rule" backwards)

Giving, $log_c\, a^n = n\, log_c\, a$ □

Illustration: Verify that, $log_2 8^2 = 2\, log_2 8$

2.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 25

Question 1

Use the rules of logs to prove that,

$$log_2 3 + 2\, log_2 5 = log_2 75$$

[3 marks]

Question 2

Use the rules of logs to prove that,

$$log_7 45 - 2\, log_7 3 = log_7 5$$

[3 marks]

Question 3

Use the rules of logs to prove that,

$$3\, log_5 3 - 2\, log_5 6 + 2\, log_5 2 = log_5 3$$

[4 marks]

Question 4

A-Level Examination Question from May 2006, paper C2, Q3 (Edexcel)

(i) Write down the value of

$$log_6 36$$

[1 mark]

(ii) Express $2\,log_a 3 + log_a 11$ as a single logarithm to base a.

[3 marks]

Question 5

A-Level Examination Question from January 2008, paper C2, Q5 (Edexcel)

Given that a and b are positive constants, solve the simultaneous equations

$$a = 3b$$

$$log_3 a + log_3 b = 2$$

Give your answers as exact numbers.

[6 marks]

Question 6

(i) Use the rules of logs to prove that,

$$3\,log_3\, x \;-\; log_3\, 7x \;+\; log_3\, 28 \;=\; 2\,log_3\, 2x$$

[3 marks]

(ii) Hence, or otherwise, prove that,

$$3\,log_2\, x \;-\; log_2\, 7x \;+\; log_2\, 28 \;=\; 2\;(\,1 \;+\; log_2\, x\,)$$

[2 marks]