8.1 The General Exponential Non-Linear Relationship Model

In Lesson 7 we investigated a situation where bivarate data, when plotted on a graph, had a relationship that was better captured by an exponential curve of best fit, rather than a straight line of best fit.

When an exponential relationship is suspected the original data should be coded using;

$$Y = \ln y \qquad \text{and} \qquad X = x$$

and a plot of Y against X made.

If the graph of the coded data is still curved, then the exponential model is NOT appropriate. Other codings could be tried to test for other types of non-linear relationship.

The jackpot has been hit, however, if the graph of the coded data is clearly linear for this then this confirms that the original data can indeed be effectively modelled by an exponential curve of the form

$$y = a\,b^x$$

and the task now becomes one of determining the values of the constants a and b. This is done by obtaining the equation of the straight line of best fit for the coded data, in the form

$$Y = A + BX$$

and using the decoding

$$a = e^A \qquad \text{and} \qquad b = e^B$$

8.2 Why Does It Work ?

If

$$y = a\,b^x$$

then

$$\ln y = \ln\left(a\,b^x\right)$$

$$\ln y = \ln a + \ln b^x$$

$$\ln y = \ln a + x \ln b$$

$$\ln y = \ln a + (\ln b)\,x$$

which is in the form of the straight line

$$Y = A + BX$$

where

$$Y = \ln y \qquad A = \ln a \qquad B = \ln b \qquad \text{and} \qquad X = x$$

the first and last give the coding, reversing the middle two gives the decoding ☐

8.3 Use of log to the base 10

It does not have to be the natural log, *ln*, that is used to code the data.
A log to any suitable base could be used.
Often, examination questions use $\log_{10}$ in which case the decoding would be

$$a = 10^A \quad \text{and} \quad b = 10^B$$

When log is written without a base in this course you should assume the base is 10.

8.4 Example

On 1st February 2020, Nick began gathering data on the relationship between the size of a population of rabbits, n, on the small Outer Hebridean Island of *Ensay*, over time, t, measured in months.
He codes his data using;

$$Y = \log n \quad \text{and} \quad X = t$$

The resulting relationship between Y and X appears to be linear with line of best fit

$$Y = 1.380 + 0.0969\,X$$

Consequently an exponential curve of best fit is the most appropriate model for the relationship between n and t
That is,

$$n = a\,b^t$$

(i) Determine the values of the constants a and b

[2 marks]

(ii) How many rabbits were there when Nick started gathering data ?

[1 mark]

(iii) How many rabbits were there one month later ?

[1 mark]

(iv) What does this model predict the rabbit population will be on the small Island of *Ensay* in 20 years time ?

[2 marks]

(v) Explain why this prediction for the number of rabbits in 20 years time should be treated with caution.

[2 marks]

8.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 30

Question 1

Use your calculator to find, correct to 3 significant figures;

(i) $10^{2.3}$ **(ii)** e^4 **(iii)** $\ln 0.4$

(iv) $\log 3.8$ **(v)** $10^{-0.2}$ **(vi)** * $\log_3 11$

[6 marks]

* HELP for Graphic Calculator fx-CG50 users

In Comp Mode (MENU 1) press EXIT then F4 (MATH) then $(\log_a b)$

Question 2

The line of best fit for some coded data passes through the points (0, 4.8) and (1.4, 7.3)
Given that the data has been coded using

$$Y = \log y \quad \text{and} \quad X = x$$

determine the relationship between y and x, writing your answer in the form

$$y = a\,b^x$$

where a and b are constants, the values of which you should find.

[3 marks]

Question 3

Data are coded using

$$Y = \log y \qquad \text{and} \qquad X = x$$

to give a linear relationship.

The equation of the line of best fit for the coded data is

$$Y = 0.8 - 0.05\,X$$

(i) Determine the relationship between y and x, writing your answer in the form

$$y = a\,b^{x}$$

where a and b are constants, the values of which you should find.

[2 marks]

(ii) Complete the following table

x	0	1	2	3
y				

[2 marks]

(iii) Sketch the curve of best fit, marking on the value of any points where the curve passes through the coordinate axes.

[2 marks]

Question 4

The table shows some data collected on the temperature, t °C, of a colony of insect larvae and the growth rate, g, of the population.

The reading for the growth rate when t is 17°C has been lost.

Temp, t °C	13	17	21	25	26	28
Growth rate, g	5.37		13.29	20.91	23.42	29.38

The data are coded using the changes of variable

$$Y = \ln g \quad \text{and} \quad X = t$$

The line of best fit of y on x is found to be

$$Y = 0.208 + 0.113\,X$$

(i) Given that the data can be modelled by an equation of the form

$$g = a\,b^{t}$$

where a and b are constants, find the values of a and b

[2 marks]

(ii) Give an interpretation of the constant b in this equation.

[1 mark]

(iii) What does the model predict the growth rate will be when the temperature is 17 °C

[1 mark]

(iv) Give an interpretation of the constant a in this equation and also explain why this constant can not be reliably checked by direct measurement.

[2 marks]

Question 5

400 identical six sided biased dice are to be rolled repeatedly.

The probability of rolling a '6' is unknown.

Every time the dice are rolled all those showing a '6' are considered ***dead*** and removed.
Those dice not yet ***dead*** are considered to be ***live***.

The table below shows the number of dice still ***live*** after the *n* th roll.

Roll number, *n*	0	1	2	3	4	5	6	7	8
N° of ***live*** dice, *A*	400	272	186	130	95	69	51	34	25

Theory predicts exponential decay and so the data are coded using

$$Y = \log L \qquad \text{and} \qquad X = n$$

The equation of the line of best fit for the coded data is

$$Y = 2.58 - 0.148\,X$$

(i) Determine the relationship between *n* and *A*, writing your answer in the form

$$L = a\,b^n$$

where *a* and *b* are constants, the values of which you should find

[3 marks]

(ii) Give an interpretation of the constant *b* in this equation.

[1 mark]

(iii) Complete the third row of the following table to show the N° of ***live*** dice that are predicted after each roll by your part **(i)** exponential curve of best fit formula.

Roll number, *n*	0	1	2	3	4	5	6	7	8
N° of ***live*** dice, *L*	400	272	186	130	95	69	51	34	25
Prediction of *L*									

[3 marks]

(iv) What is the probability of rolling a '6' with these biased dice.
Justify your answer.

[2 marks]