A-Level Mathematics

Year 2

~ Pure Mathematics ~

F U N C T I O N S
I I

The Loch Ness Monster Function

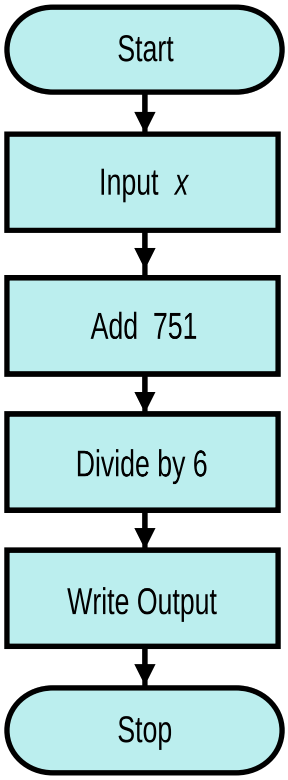
$$y = 1 + \sqrt{\cos x}$$

F U N C T I O N S   I I

1.1 What is a function ?

A basic electronic calculator is a good example of a function machine.
A single number is input, buttons are pressed, and a single number output results.

Here is a flow chart of a function.



If the number 149 is input, what is the output ?

[1 mark]

The function, f, described by the flow chart is;

$$f(x) = \frac{x + 751}{6}$$

Determine the value of, $f(-151)$

[1 mark]

1.2 Domain and Range

The set of numbers allowed into a function is referred to as the function's ***domain***.
Typically, this will be as big as possible, to maximise the function's usefulness.
The set of numbers that can come out of a function is the function's ***range***.
Often, a function's domain is given.
Sometimes, but not often, a function's range is also given.

1.2.1 Example

$$\underbrace{f(x) = \frac{12}{6 - x},}_{\text{function}} \quad \underbrace{x \in \mathbb{R},\ x \neq 6,}_{\substack{\text{domain} \\ \text{(numbers allowed in)}}} \quad \underbrace{f(x) \in \mathbb{R},\ f(x) \neq 0}_{\substack{\text{range} \\ \text{(numbers coming out)}}}$$

In this example, all real numbers except 6 may be used as the input number.
The 6 is excluded because is would result in a division by zero which is not defined.
All real numbers are possible outputs, except no input will give 0 as output.

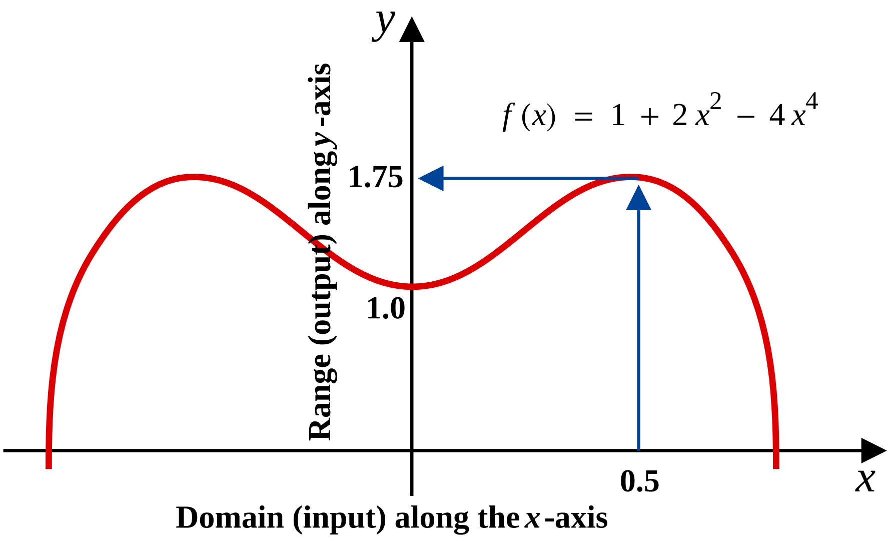
1.2.2 Example

$$f(x) = \sqrt{x - 4}, \qquad x \in \mathbb{R},\ x \geqslant 4, \qquad f(x) = \mathbb{R},\ f(x) \geqslant 0$$

In this example, all real numbers greater than or equal to 4 may be input.
Numbers less than 4 are excluded to avoid the square rooting of a negative number.
All real numbers greater than or equal to 0 are possible outputs.

1.3 Graphs of functions

A graph, often a quick sketch of $y = f(x)$, is a good visualisation of a function.
The input numbers, the domain, are along the x-axis.
The output numbers, the range, are along the y-axis.



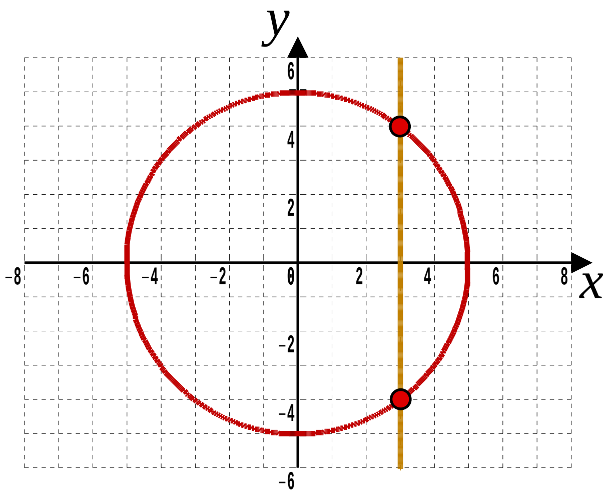
The sketch shows 0.5 going into the function and 1.75 coming out.

1.4 Is it a Function ?

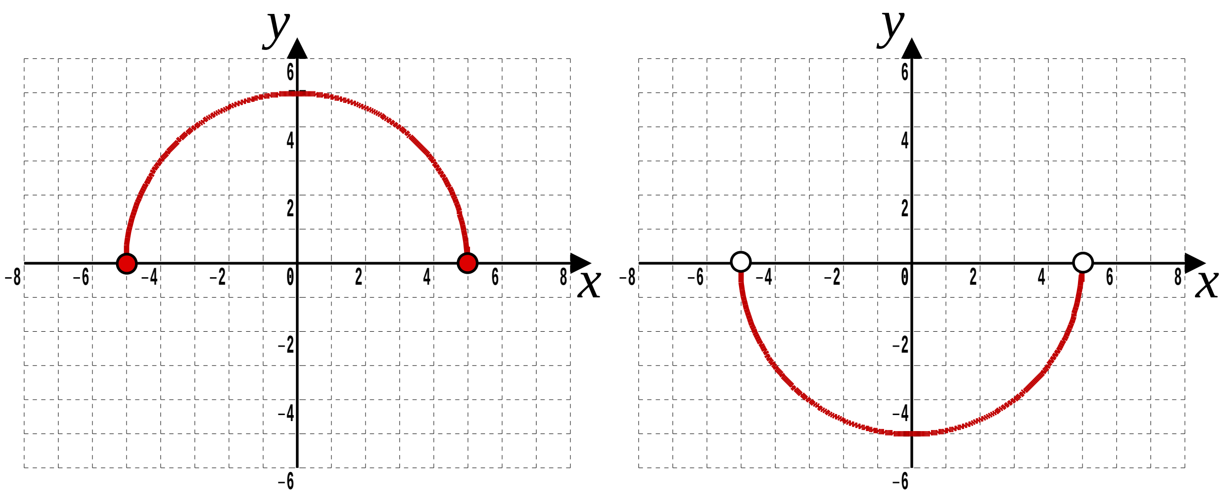
When looking at the graph of a piece of algebra, to be a function no vertical line must intersect the graph more than once.

1.4.1 Example

No entire circle can be a function because a vertical line can be found that intersects the curve more than once.



The circle $x^2 + y^2 = 5^2$ is not a function because a vertical line, for example $x = 3$, can be found that cuts its graph more than once.



However, that circle can be broken up into two separate pieces each of which is a function. Notice that the end points have been allocated to the piece depicted on the left hand graph but they could both have been allocated to the right hand graph or one given to each. In breaking up the circle, it has been made clear where they went. Here are the two semicircle functions, graphed above;

$$f(x) = \sqrt{5^2 - x^2} \qquad x \in \mathbb{R}, \quad -5 \leqslant x \leqslant 5 \qquad \text{(Left hand graph)}$$

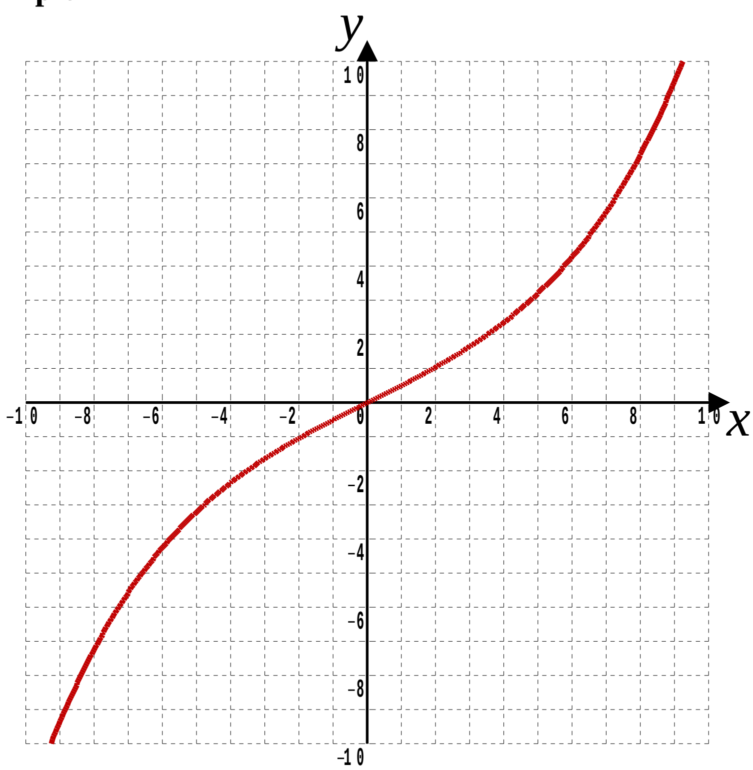
$$g(x) = -\sqrt{5^2 - x^2} \qquad x \in \mathbb{R}, \quad -5 < x < 5 \qquad \text{(Right hand graph)}$$

Notice how the endpoint allocation is captured by the inequalities of the domains.

1.5 One-to-One functions

A function is one-to-one if every input has a unique output, and vice-versa.
On the graph of the function, every horizontal line will cut the graph exactly once.

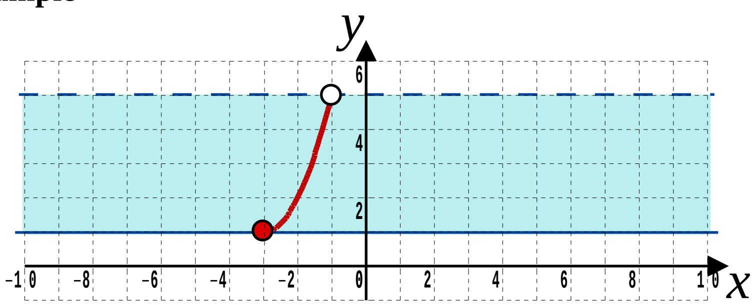
1.5.1 Example



The graph is of the function, $f(x) = e^{\frac{x}{4}} - e^{-\frac{x}{4}} \quad x \in \mathbb{R}$

The function is one-to-one because a horizontal line, no matter where it is drawn, will intersect the graph of this function exactly once.

1.5.2 Example



The graph is of the function,

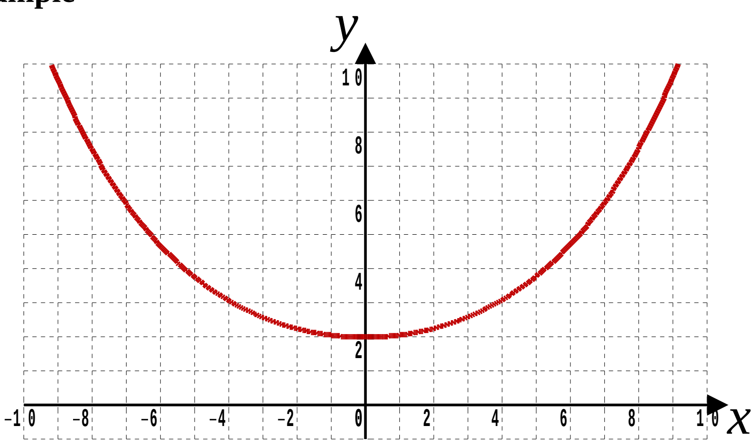
$f(x) = (x+3)^2 + 1, \quad x \in \mathbb{R}, \quad -3 \leqslant x < -1, \quad f(x) \in \mathbb{R}, \quad 1 \leqslant f(x) < 5$

The function is one-to-one because a horizontal line, no matter where it is drawn, will intersect the graph of this function exactly once, within the range (shaded blue) of the function.

1.6 Many-to-One Functions

A function is many-to-one if more than one input maps to the same output.
Graphically, a horizontal line can be found that will cut the plot of the function more than once.

1.6.1 Example



The graph is of the function, $f(x) = e^{\frac{x}{4}} + e^{-\frac{x}{4}} \quad x \in \mathbb{R}$

The function is many-to-one because a horizontal line can be found that will intersect, the graph of this function more than once.
Add your own example of such a line to the graph above.

[1 mark]

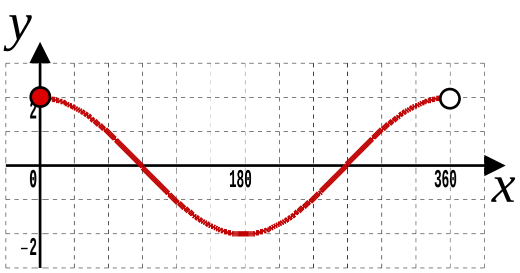
1. 7 Breaking Up Functions

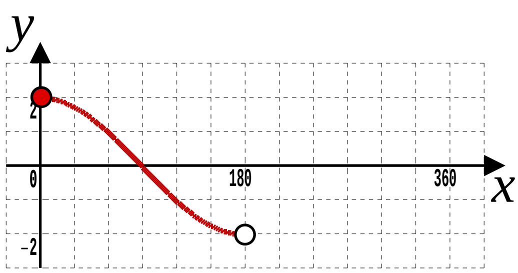
Mathematicians like one-to-one functions because they have many special properties.
For example, only a one-to-one function has an inverse.
(This key result will be explored later, in lesson 7)
The desire to have one-to-one functions results in many-to-one functions being broken up into pieces, each of which is one-to-one.

1.7.1 Example

Consider the function, $f(x) = \cos x, \ x \in \mathbb{R}, \ \ 0 \leqslant x < 360°$

The graph of $y = f(x)$ shown left, reveals that this function in many-to-one.
The right hand graph shown the one-to-one piece that a calculator uses to give values for the inverse cosine function, *arccos x*





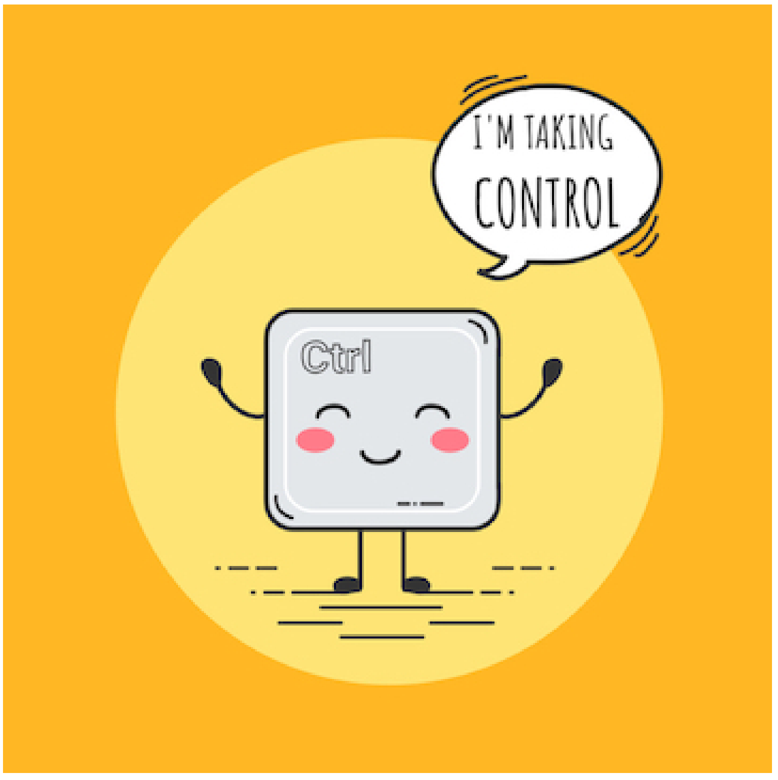
The right hand one-to-one graph is $g(x) = \cos x, \ x \in \mathbb{R}, \ \ 0 \leqslant x < 180°$

1.8 The Bigger Picture

The topic of Functions may seem technical and awkward at first. The driving force behind thinking about them is this pedantic way is summed up by “Taking Control”. Being able to specify a domain allows mathematicians to shut out confusions caused by such things as “division by zero” and the “taking of negative square roots”.
(Later on, negative square roots get let back in but under tight control !)
As a further example, rather than always letting all real numbers in, there are situations where only the integers are to be worked with.

The simple statement in the domain, $x \in \mathbb{Z}$, shuts out all but the integers.

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$



1.9 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 60

Question 1

(i) Sketch the graph of the squaring function, $f(x) = x^2, \; x \in \mathbb{R}$

[2 marks]

(ii) Show that the squaring function is many-to-one by adding a horizontal line to your sketch that cuts the graph twice.

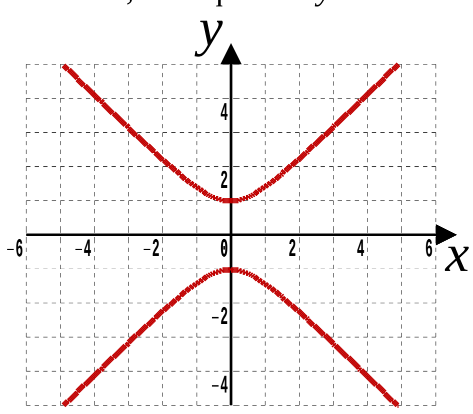
[1 mark]

(iii) What is the range of the squaring function ?

[2 marks]

Question 2

The hyperbola, graphed below, has equation $y^2 - x^2 = 1$



Explain why this cannot be the graph of a function.

[2 marks]

Question 3
State a value of x that must be excluded from the domain of the following function;

$$f\,(x) \;=\; \frac{1}{(\,1 \,+\, x\,)^2}$$

Give a reason for your answer.

[2 marks]

Question 4

(i) Sketch the graph of $f\,(x) \;=\; \cos x, \quad x \;\in\; \mathbb{R}, \quad 0 \;\leqslant\; x \;<\; 360°$

[3 marks]

(ii) Show that the function is many-to-one by adding a horizontal line to your sketch that cuts the graph more than once.

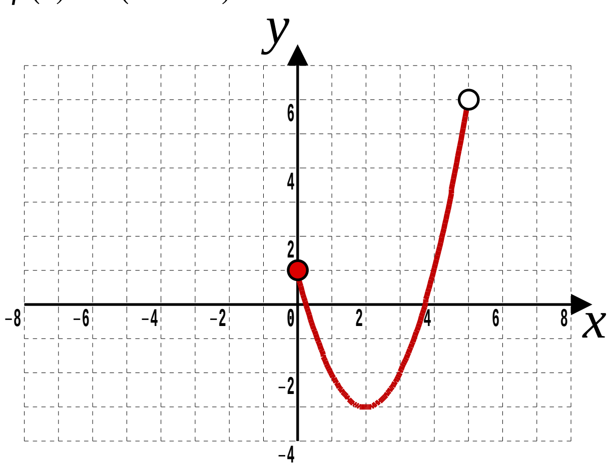
[1 mark]

(iii) What is the range of the cosine function ?

[2 marks]

Question 5

The graph is of $f\,(x) \;=\; (\,x \,-\, 2\,)^2 \,-\, 3$ but with a restricted domain.



Taking care over any inequalities, write down the restricted domain of this function.

[2 marks]

Question 6

(i) Sketch the graph of $f(x) = \tan x$ over the interval $0° \leqslant x \leqslant 360°$

[2 marks]

(ii) Show that the tangent function is many-to-one by adding a to your sketch a horizontal line that cuts the graph more than once.

[1 mark]

(iii) State the values of x that must be excluded from the domain of $f(x) = \tan x$ over the interval $0° \leqslant x \leqslant 360°$
Give a reason for your answer.

[3 marks]

Question 7

(i) Sketch the cubing function, $f(x) = x^3, \quad x \in \mathbb{R}$

[2 marks]

(ii) Is the cubing function many-to-one or one-to-one ?
Give a reason for your answer.

[2 marks]

Question 8

Here is the description of a hybrid function made from two pieces.

$$f(x) = \begin{cases} \sqrt{x} & 0 \leqslant x < 1 \\ 2 - x & 1 \leqslant x \leqslant 4 \end{cases}$$

This has domain $x \in \mathbb{R},\ 0 \leqslant x \leqslant 4$

(i) Sketch the graph of $f(x)$

[3 marks]

(ii) Show that $f(x)$ is many-to-one by adding a horizontal line to your sketch that cuts the graph more than once.

[1 mark]

(iii) State the range of $f(x)$

[3 marks]

Question 9

State the range of the function

$$f(x) = 3 + \cos x, \quad x \in \mathbb{R}$$

(A sketch may help)

[3 marks]

Question 10

Consider the function, $f(x) = x^2 - 2x + 9, \qquad x \in \mathbb{R}$

(i) Express $f(x)$ in the completed square form.

That is, in the form $f(x) = (x - a)^2 + b$

for some constants a and b which are to be determined.

[2 marks]

(ii) Hence, or otherwise, sketch the graph of $f(x)$ paying particular attention to the coordinates of the minimum point.

[2 marks]

(iii) Hence, or otherwise, state the range of the function.

[2 marks]

Question 11

$$f(x) = \sqrt{1 - 4x}$$

(i) State the domain of $f(x)$, assuming it is as large as possible.

[2 marks]

(ii) State the range of $f(x)$.

[2 marks]

Question 12

$$f(x) = (x - 2)^2 + 5, \qquad x \in \mathbb{R}, \qquad 1 \leqslant x \leqslant 5$$

(i) Sketch $f(x)$ paying particular attention to the vertex of the parabola and the points at either end of the closed interval on which it is defined.

[3 marks]

(ii) State the range of $f(x)$

[2 marks]

(iii) Solve, $f(x) = 9$

[2 marks]

Question 13

A-Level Examination Question from January 2019, paper C3, Q3(a) (Edexcel)

The function f is defined by,

$$f: x \to 2x^2 + 3kx + k^2 \quad x \in \mathbb{R}, \quad -4k \leqslant x \leqslant 0$$

where k is a positive constant.

Find, in terms of k, the range of f

[6 marks]