3E.1 Rational Functions and Asymptotes

The asymptotes of rational functions can be of three types; horizontal, vertical or oblique. Behind the scenes, as with differentiation, the theory of limits is at play. However, as with differentiation, a set of straight forward rules finds the asymptotes without having to battle through tedious technicalities.

Definition : A Rational Function...

...is of the form $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

In such functions the degree of the two polynomials, $P(x)$ and $Q(x)$ is of crucial importance when determining the asymptotes.
With each polynomial arranged with the powers of x in decreasing order the leading term is then the first term.
The coefficient of the leading term of $P(x)$ is p
The coefficient of the leading term of $Q(x)$ is q

That is,

$$P(x) = p\,x^{deg(P(x))} + other\ lower\ power\ terms$$

$$Q(x) = q\,x^{deg(Q(x))} + other\ lower\ power\ terms$$

Test for Horizontal Asymptotes

Compare the degree of the numerator $P(x)$ with that of the denominator $Q(x)$;

If $deg(P(x)) > deg(Q(x))$ then there is no horizontal asymptote.

If $deg(P(x)) = deg(Q(x))$ then $y = \dfrac{p}{q}$ is a horizontal asymptote.

If $deg(P(x)) < deg(Q(x))$ then the x-axis is a horizontal asymptote.

Test for Vertical Asymptotes

Set the denominator, $Q(x)$, equal to zero and solve.

The solutions are the equations of vertical asymptotes.
(Unless the numerator has an identical solution in which case there is a point hole in the curve for that value of x)

Test for Oblique Asymptotes

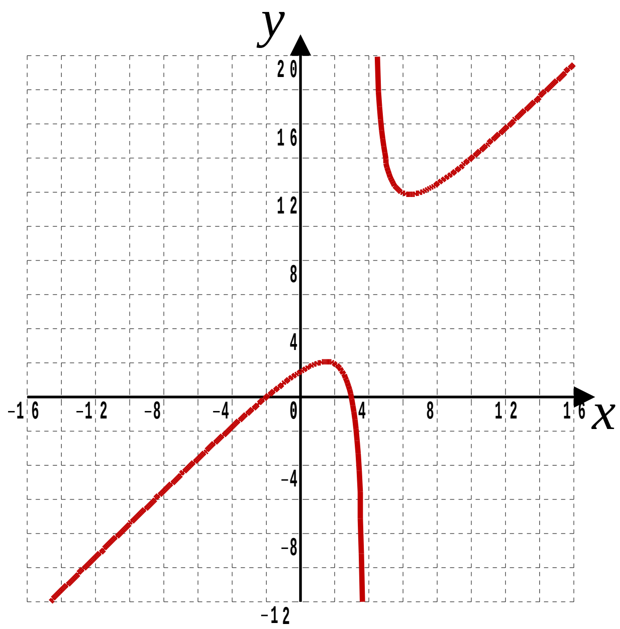
If $deg(P(x)) - deg(Q(x)) = 1$ then there exists an oblique asymptote.

Polynomial division of $P(x)$ by $Q(x)$ gives its equation.
(Any remainder from the division is ignored)

3E.2 Example

Find all asymptotes to the curve, $y = \dfrac{x^2 - x - 6}{x - 4}$

Once found, carefully add them to the graph of the curve presented below.



A full solution is presented on the following page.

[6 marks]

3E.3 Solution

$$P(x) = x^2 - x - 6 \; : \; deg(P(x)) = 2$$

$$Q(x) = x - 4 \; : \; deg(Q(x)) = 1$$

Testing for horizontal asymptotes;

As $deg(P(x)) > deg(Q(x))$ there is no horizontal asymptote.

Testing for vertical asymptotes;

$$Q(x) = 0$$

$$x - 4 = 0$$

$$x = 4$$

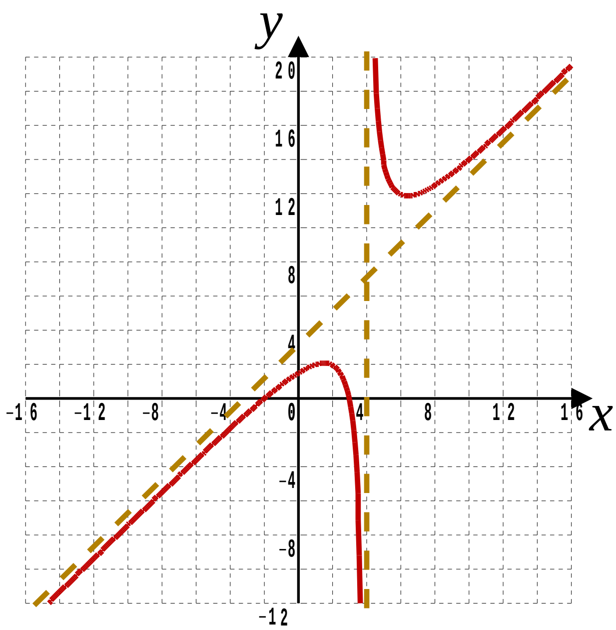
There is a vertical asymptote with equation, $x = 4$

Testing for oblique asymptotes;

As $deg(P(x)) - deg(Q(x)) = 1$ there exists an oblique asymptote.

$$\begin{array}{r|rrr} & x & + & 3 \\ \hline x - 4 & x^2 - & x & - \; 6 \\ & x^2 - & 4x & \\ \hline & & 3x & - \; 6 \\ & & 3x & - \; 12 \\ \hline & & & 6 \end{array} \quad \leftarrow \text{ Ignore !}$$

There is an oblique asymptote with equation $y = x + 3$



[6 marks]

3E.4 Exercise

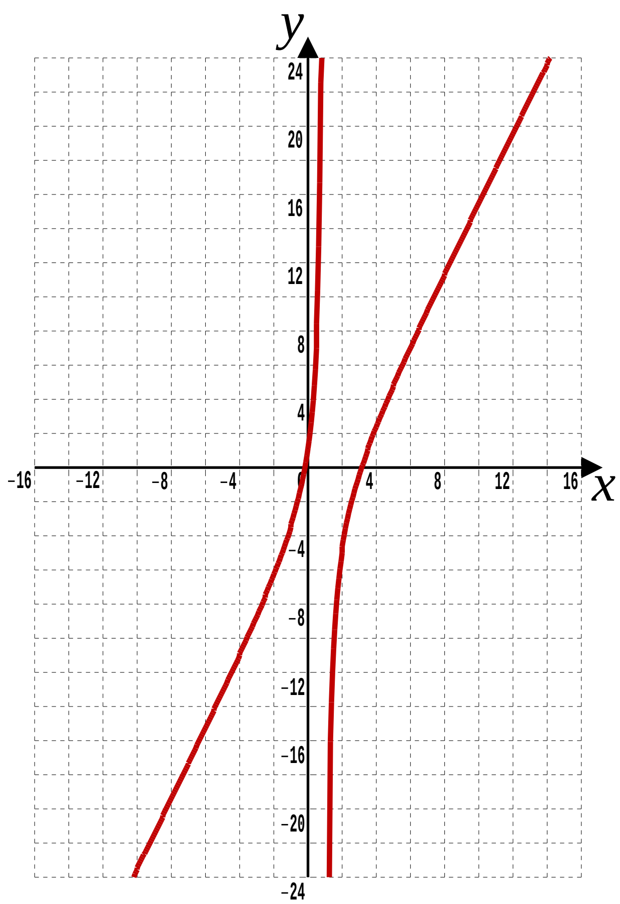
Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 18

Question 1

Find all asymptotes to the curve, $y = \dfrac{2x^2 - 6x - 1}{x - 1}$

Once found, carefully add them to the graph of the curve presented below.

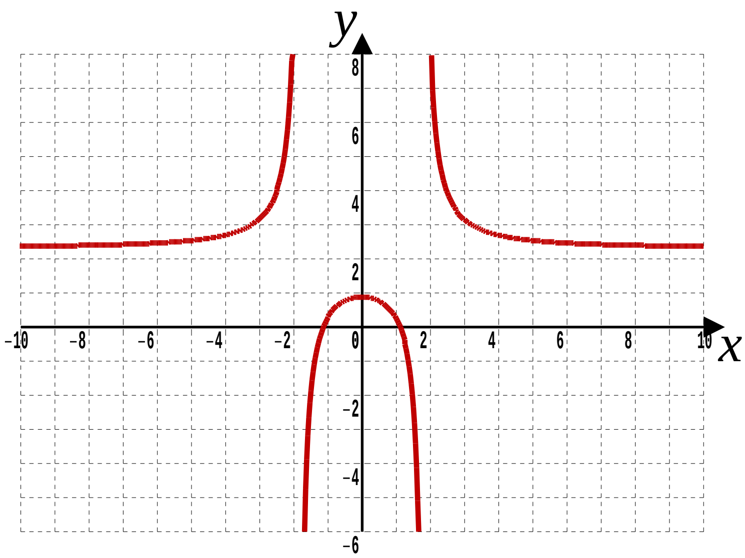


[6 marks]

Question 2

Find all asymptotes to the curve, $y = \dfrac{7x^2 - 9}{3x^2 - 10}$

Once found, carefully add them to the graph of the curve presented below.

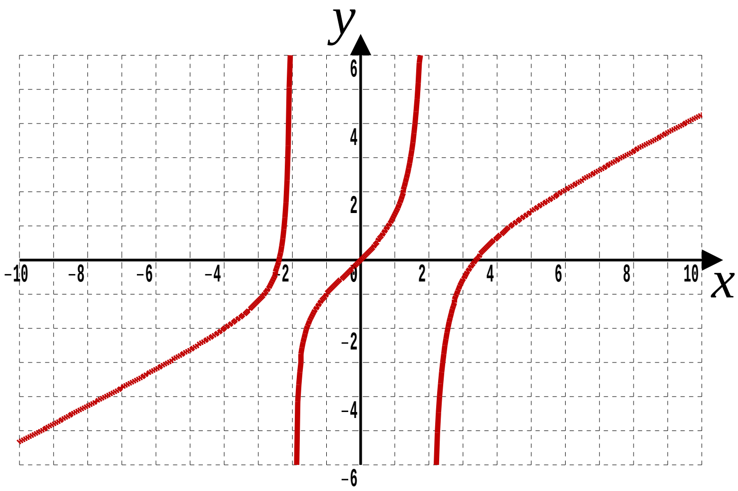


[6 marks]

Question 3

Find all asymptotes to the curve, $y = \dfrac{x^3 - x^2 - 8}{2x^2 - 8}$

Once found, carefully add them to the graph of the curve presented below.



[6 marks]