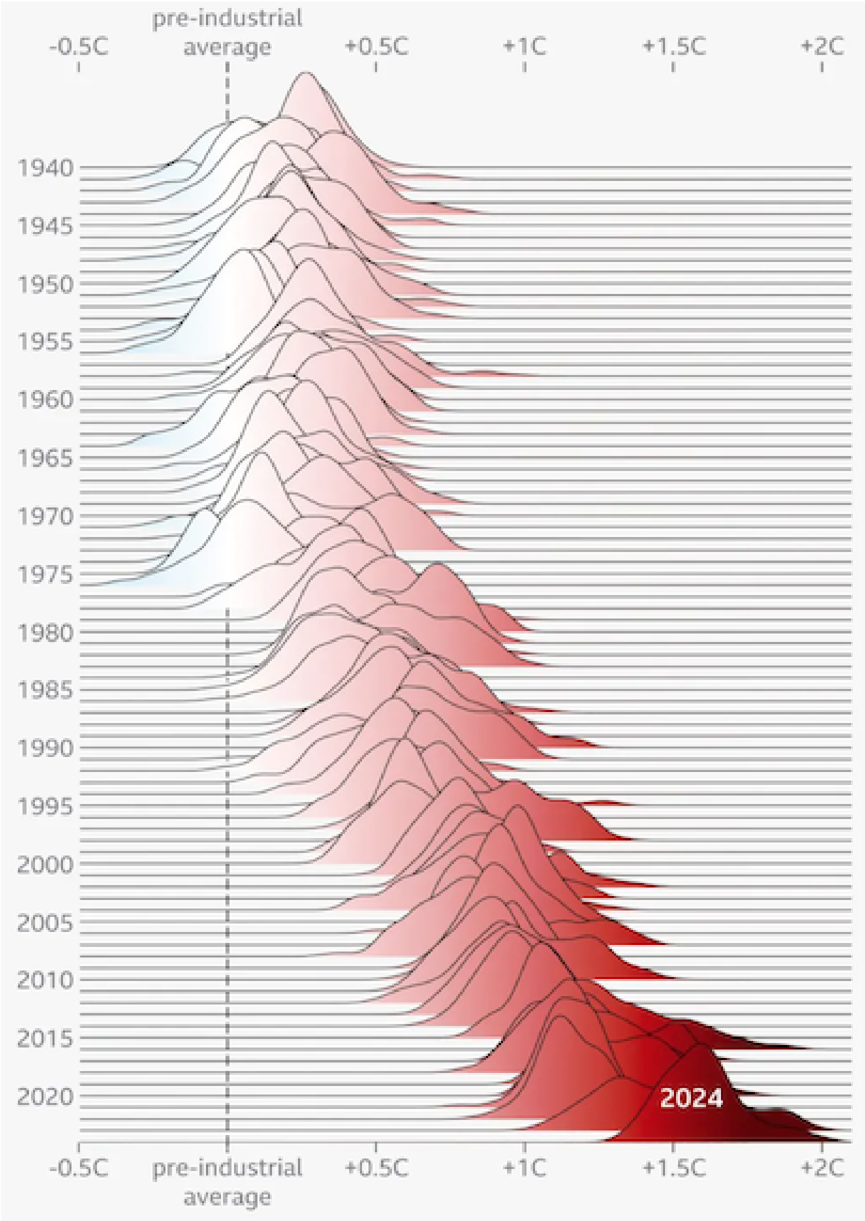
A-Level Applied Mathematics
~ Statistics ~
Year 1

HYPOTHESIS TESTING



The distribution of daily global air temperature differences from the pre-industrial average (1850-1900), 1940-2024.
BBC News on 10th January 2025
(Their source: ERA5, C3S/ECMWF)

Hypothesis Testing

Lesson 1

A-level Applied Mathematics
Statistics : Hypothesis Testing : Year 1

1.1 Handling An Unknown Probability

A company produces a mechanical component and monitors the number of defectives that roll off the production line by testing a sample of twenty every hour. The number of defectives in the sample over the course of time will form a large ***population*** of values such as 2, 0, 1, 6, 3, 4, 0, 3, 3, ...

The company may decide to model the distribution of the defectives, X, as;

$$X \sim B(20, p)$$

but it cannot be sure of the value of p.

We call p a ***population parameter***. There may be several parameters of interest in the population. For example, in this population we might also be interested in the standard deviation of the number of defectives in the sample. Although the company may not be able to find out the exact value of p it may wish to make ***inferences*** about it. Statisticians achieve this in a mathematically rigorous manner by carrying out a ***hypothesis test***.

1.2 Setting Up A Hypothesis Test

A hypothesis is a statement made about the value of a population parameter. In the binomial case, this is about p. Typically, the hypothesis is then tested, often by taking a sample from the population as described above, or by performing an experiment, such as tossing a coin 50 times. The result from the sample, or experiment, is called a ***test statistic***. So, for example, the numbers from the situation above, 2, 0, 1, 6, 3, 4, 0, 3, 3, ... are all test statistics; the results of looking for the number of defectives in samples of size 20. With the coin, obtaining 27 'tails' would be a test statistic.

In order to implement a hypothesis test you first need to clearly state ***two*** hypotheses about p. These are;

◇ ***The null hypothesis***, H_0

This is the hypothesis about p that is assumed to be correct.

◇ ***The alternative hypothesis***, H_1

This is a fallback statement about p, true if H_0 is shown to be false.

1.3 Example

Peter wants to know if a dice is fair or if it is biased against rolling a "6".

He rolls the dice 60 times and counts the number of times that it rolls "6".

(i) Describe the test statistic.

(ii) Write down a suitable null hypothesis.

(iii) Write down a suitable alternative hypothesis.

1.4 The One Tailed Hypothesis Test (Low End Tail)

In Peter's case, because he was looking for a bias ***against*** rolling a "6", the alternative hypothesis was $H_1 : p < \frac{1}{6}$

This is an example of a one tailed test.

Had Peter been looking for a bias in ***favour*** of rolling a "6" the alternative hypothesis would still has been a one tailed test. However, the inequality would go the 'other way'. That is, $H_1 : p > \frac{1}{6}$

In a later lesson the two tailed test will be considered but, for now, simply note that in Peter's case a two tailed test would had resulted from him thinking that the alternative hypothesis was $H_1 : p \neq \frac{1}{6}$

All of this lesson's questions are about one tailed tests.

1.5 Example

Back with Peter and his suspicion that his dice is biased against rolling a "6", his null hypothesis is $H_0 : p = \frac{1}{6}$ with alternative hypothesis $H_1 : p < \frac{1}{6}$

From 60 rolls he is expecting 10 rolls of "6".

Suppose that he obtains 7 rolls of a six.

The question to now ask is this: “Could this lower than expected number of "6" rolls simply be an unlucky occurrence but one that is still consistent with p being $\frac{1}{6}$ (H_0) or is it so many fewer than the ten expected that it's reasonable to conclude that the probability, p, of rolling a "6" is indeed less than $\frac{1}{6}$ (H_1) ?”

1.6 Exercise

Question 1

A coin is believed to be biased with a probability of landing 'heads' of 0.4
The coin is to be investigated by tossing it 50 times.

(i) How many times is 'heads' expected to occur ?

(ii) In the investigation, less heads than expected are obtained, only 12.
Using this observation, test $H_0 : p = 0.4$ against $H_1 : p < 0.4$
Use a 5 % significance level in deciding if to reject H_0 or not.

You may use a calculator or printed tables to look up values of the Binomial Cumulative Distribution Function (BCDF) for $X \sim B(50, 0.4)$

Question 2

A single observation, x, is taken from a binomial distribution B(25, 0.35).
A value of $x = 4$ is obtained.

(i) Is this value higher or lower than that expected ?

(ii) Use this observation to test $H_0 : p = 0.35$ against $H_1 : p < 0.35$ using a 5 % significance level.

(iii) Have you rejected H_0 or not ?

Question 3

A random variable, X, is thought to have distribution $X \sim B(22, 0.72)$

A single observation, x, is to be taken from this distribution.

(i) What is the expected value of x ?

The single observation is lower than expected, $x = 12$.

(ii) Using your statistics calculator, test at the 5 % significance level, $H_0 : p = 0.72$ against $H_1 : p < 0.72$

Clearly state if, in conclusion, you reject H_0 or not.

Question 4

A coin is to be tossed 30 times.

Lucy suspects that it is biased against landing tails.

(i) Using the model, $X \sim B(30, 0.5)$ and hypothesis test $H_0\ \ p = 0.5$, against $H_1 : p < 0.5$ determine the ***highest*** number of tails that would cause the null hypothesis to be ***rejected*** at the 5 % significance level.

HINT : Easiest done using the printed BCDF tables

The ***critical region*** is the region of the probability distribution which, if the test statistic falls within it, would cause the null hypothesis to be rejected.

(ii) State the critical region for question 4

The ***critical value*** is the first value to fall inside of the critical region.

(iii) State the critical value for question 4

Question 5

A dice is to be rolled 60 times.

Robert suspects that it is biased against rolling a six.

(i) Using the model, $X \sim B(60, \frac{1}{6})$ and hypothesis test H_0 $p = \frac{1}{6}$, against $H_1 : p < \frac{1}{6}$ determine the ***highest*** number of sixes that would cause the null hypothesis to be ***rejected*** at the 5 % significance level.

NOTE : Easiest done using the printed BCDF tables **BUT** tables don't exist for $n = 60$ or $p = \frac{1}{6}$ so you'll have to use you calculator.

The ***critical region*** is the region of the probability distribution which, if the test statistic falls within it, would cause the null hypothesis to be rejected.

(ii) State the critical region for question 5

The ***critical value*** is the first value to fall inside of the critical region.

(iii) State the critical value for question 5

Question 6

A random variable, X, has a distribution $X \sim B(40, 0.3)$.

With $H_0 : p = 0.3$ against $H_1 : p < 0.3$ and using a 5 % level of significance, find the critical region of this test.