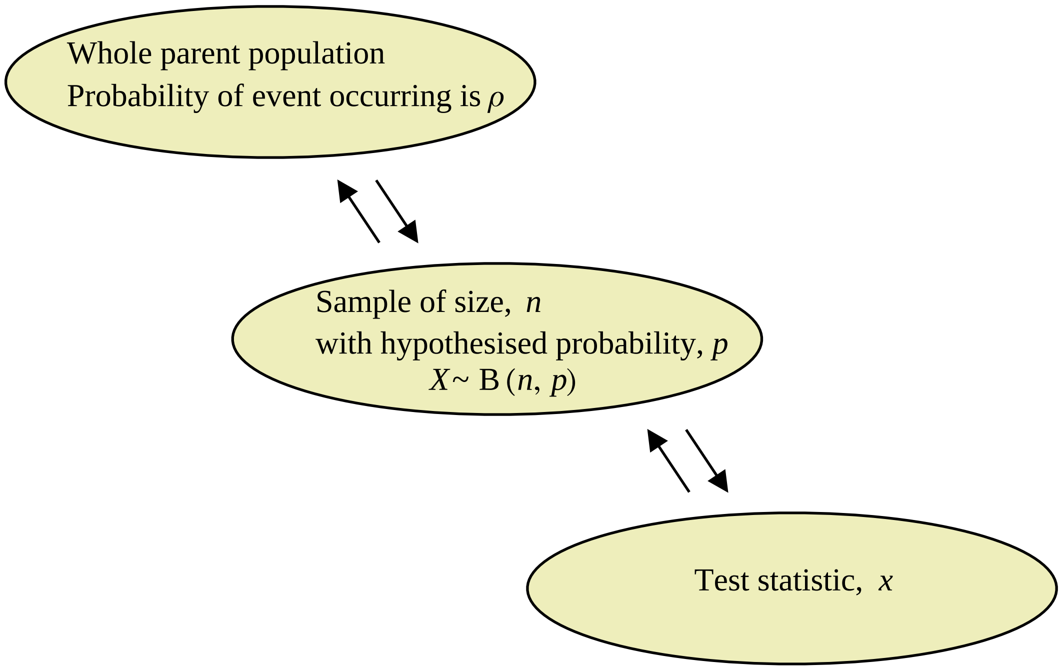
Lesson 3

3.1 Grasping A Bigger Picture

We have been looking at situations where a whole population is binomially distributed and in which the probability of an event of interest occurring is ρ. For example, this could be the infinite number of possible rolls of a “5” on a biased dice. We do not know the value of ρ. From that whole population a sample of a known size n is taken and we hypothesis on what the value of the probability of rolling a “5” in this sample, p, might be. (We may well start by assuming it's one sixth). Then we actually roll the dice n times; an experiment of n trials is performed. From this a test statistic, x, is obtained. By the technique of hypothesis testing on $X \sim B(n, p)$, we seek to determine if the test statistic, x, is acceptably consistent with the suggested value of p in the sample population and, therefore (assuming it's a representative sample), consistent with ρ in the whole 'parent' population.



Lets recap the sort of question tackled previously before looking at it's shortcoming (of not seeing the bigger picture).

As an example, given the binomial distribution $X \sim B(15, 0.45)$, the test statistic $x = 2$ and assuming a one tail (low end) test with significance level 5 %, is the test statistic consistent with p being 0.45 ?

A typical solution would be;

$H_0 : p = 0.45, \quad H_1 : p < 0.45$

$P(X \leqslant 2) = 0.0107$ (from tables or a calculator)

As $1.07\% < 5\%$, H_0 is rejected

Statistical evidence suggests that $p < 0.45$

3.2 The Critical Region and The Acceptance Region

The approach outlined by the above example is all very well, in that it answered the question, but a grasp of the (slightly) bigger picture is missing. To see the 'bigger picture', a few key ideas (introduced in Exercise 2.4) need to be in place;

◇ The ***critical region***

This is the region of the probability distribution which, if the test statistic falls within it, would cause the null hypothesis to be rejected.

◇ The ***critical value***

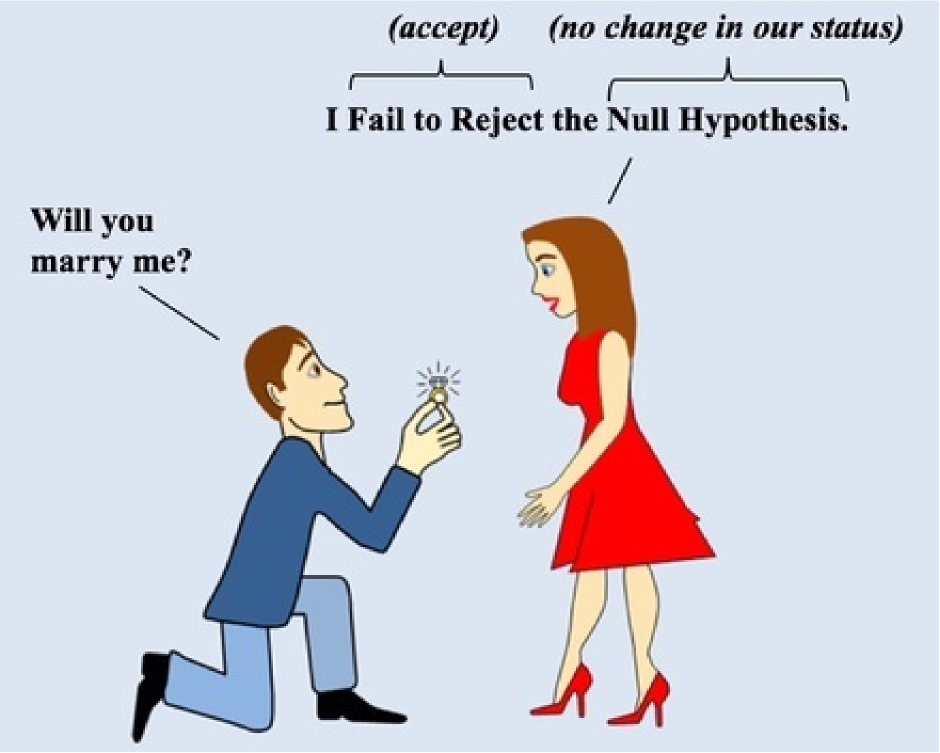
This is the first value to fall inside of the critical region.

◇ The ***acceptance region***

This is the region of the probability distribution which, if the test statistic falls within it, would cause the null hypothesis to be accepted.

Examination questions often ask for this "bigger picture' first. Such a question will first ask for the critical region or critical value. Only after that has been established will the question reveal a test statistic and ask if that causes the rejection or acceptance of the null hypothesis.

The printed Binomial Cumulative Distribution Function tables are the easier way of locating a critical region although a calculator, with more effort, can be used to obtain the same result. Sometimes a calculator is essential because the question involves a value of n or a value of p not amongst those given in the printed tables.



3.3 Example (Low Tail)

A population is believed to be binomially distributed $X \sim B(15, 0.45)$,

with $H_0 : p = 0.45$, $H_1 : p < 0.45$

(i) Determine the critical region at the 5 % significance level using the following extract from the Binomial Cumulative Distribution Function tables.

x	P(*X* = *x*)	$X \sim B(15, 0.45)$
0	0.0001	
1	0.0017	⇐1^{st} below 1 %
2	0.0107	
3	0.0424	⇐ 1^{st} below 5 %
4	0.1204	
5	0.2608	
6	0.4522	
7	0.6535	
8	0.8182	
9	0.9231	
10	0.9745	⇐ 1^{st} above 95 %
11	0.9937	⇐ 1^{st} above 99 %
12	0.9989	
13	0.9999	
14	1.0000	

(ii) State the critical value.

(iii) Given a test statistic, $x = 2$, state with a reason if the null hypothesis should be rejected or not.

(iv) State the actual significance level of this test.

3.4 The Actual Significance Level

The actual significance level of a hypothesis test is the probability of incorrectly rejecting the null hypothesis.

3.5 Example (High Tail)

A population is believed to be binomially distributed $X \sim B(15, 0.45)$,

with $H_0 : p = 0.45$, $H_1 : p > 0.45$

(i) Determine the critical region at the 5 % significance level using the previously given extract from the Binomial Cumulative Distribution Function tables.

(ii) State the critical value.

(iii) Given a test statistic, $x = 10$, state with a reason if the null hypothesis should be rejected or not.

(iv) State the actual significance level of this test.

3.6 Exercise

Question 1

A test statistic has a distribution $B(12, p)$.

(i) Given that $H_0 : p = 0.25$, $H_1 : p > 0.25$, use the printed tables to find the critical region for the test using a 5 % significance level.

(ii) State the actual significance level of the test.

Question 2

A random variable has a distribution B(20, p).

A single observation is used to test $H_0 : p = 0.3$, $H_1 : p < 0.3$

(i) Using the printed tables and a 5 % significance level, find the critical region for the test.

(ii) State the actual significance level of the test.

Question 3

A test statistic has a distribution B(18, p).

(i) Given that $H_0 : p = 0.32$, $H_1 : p > 0.32$, use your statistics calculator to find the critical region for the test using a 5 % significance level.

(ii) State the actual significance level of the test.

Question 4

Seedlings come in trays of 36.

On average, 12 seedlings survive to be plated on.

A gardener decides to use a new fertiliser on the seedlings which she believes will improve the number that survive.

(i) Describe the test statistic and state suitable null and alternative hypotheses.

[3 marks]

(ii) Using a 10 % level of significance, find the critical region for a test to check her belief.

[3 marks]

(iii) State the probability of incorrectly rejecting H_0 using this critical region.

[1 mark]

(iv) In a tray that has been treated with the new fertiliser, 19 seedlings survive. On the basis of this test statistic, state with a reason if the null hypothesis should be rejected or not and also what this tells you about the new fertiliser.

[3 marks]

Question 5

A mechanical component fails, on average, 7 times out of every 20.
An engineer designs a new system of manufacture that he believes reduces the likelihood of failure.
He tests a sample of 30 components using his new system.

(i) Describe the test statistic

[1 mark]

(ii) State suitable null and alternative hypotheses

[2 marks]

(iii) Using a 5 % level of significance, find the critical region for a test to check his belief, ***ensuring the probability is as close as possible to 0.05***

[3 marks]

(iv) Write down the actual significance level of the test.`

[1 mark]