Lesson 2

2.1 Building The Poisson Distribution

The Poisson distribution comes from a manipulation of the remarkable and beautiful infinite M[c]Laurin series expansion of e^x

$$e^{\lambda} = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^r}{r!} + \ldots$$

which is convergent for all real values of λ, and where e = 2.718281 828459 0452...

Both sides are simply multiplied by $e^{-\lambda}$ which yields,

$$1 = \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} + \ldots + \frac{e^{-\lambda}\lambda^r}{r!} + \ldots$$

The terms on the right hand side can be considered to be the probabilities of a discrete random variable X that takes the values 0, 1, 2, 3, 4, ... not least because we have arranged that the sum of the probabilities sum to 1.

Thus the Poisson distribution is obtained,

If $X \sim Po(\lambda)$, then the Poisson distribution is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, 3, \ldots$$

$$\lambda \textit{ is the rate per a specified, fixed interval}$$

2.2 When the Poisson Distribution is Appropriate

The Poisson distribution is used to model the number of times, X, that a particular event occurs within a given interval of time or space.

The events must occur,

- ◇ independently
- ◇ singly in continuous space or time
- ◇ at a constant average rate such that the mean number in the interval is proportional to the length of the interval
  (such events are called *random* or *rare* events)

The parameter, λ, is the average number of times the event occurs in a single interval

2.3 Poisson Distribution Examples

- The number of trains stopping at King's Cross platform 7 every twenty minutes
- The number of times George's heart beats per minute
- The number of spelling mistakes a typist makes per five minutes

2.4 Exercise

Question 1

Spam emails arrives in my inbox at random at an average rate of 5 a day.

(a) Give two reasons why a Poisson distribution might be a suitable model for the number of spam emails received per day.

(b) State the probability that in a randomly chosen day I receive,

(i) 3 spam emails

(ii) less than 5 spam emails

(iii) not more than 7 spam emails

Question 2

The number of misprints in the first draft of a book occur randomly at an average rate of 2.3 per page.

Determine the probability that a randomly chosen page contains 4 misprints.

Question 3

Give a reason why the number of cars passing under a motorway bridge during a traffic jam should not be modelled with a Poisson distribution.

Question 4

The number of accidents per week at a factory is a Poisson random variable with parameter 2.5.

Find the probability that

(i) exactly 5 accidents will occur in a week

(ii) more than 14 accidents will occur in 4 weeks

Question 5

S2 Examination Question from June 2017

A company secretary receives telephone calls at random at a mean rate of 2.5 calls every hour.

(a) Find the probability that the secretary receives

(i) at least 4 telephone calls in the next hour

[2 marks]

(ii) exactly 3 telephone calls in the next 15 minutes

[3 marks]

(b) Find, to the nearest minute, the maximum length of time the telephone can be left unattended so that the probability of missing a telephone call is less than 0.2

[3 marks]

The company puts an advert in the local newspaper.

The number of telephone calls received in a randomly selected 2 hour period after the paper is published is 10.

(c) Test at the 5% level of significance whether or not the mean rate of telephone calls has increased. State your hypotheses clearly

[5 marks]

Question 6

A shop sells a particular make of mobile phone at a rate of 4 per week on average.
The number sold in a week has a Poisson distribution.

(a) Find the probability that the shop sells at least 2 in a week.

(b) Find the smallest number that can be in stock at the beginning of a week in order to have at least a 99% chance of being able to meet all demands during that week

Question 7

It is estimated that the probability that a student will be cheating in an examination is 0.5%.
An examination is being sat by 120 students.
Determine the probability that at least 2 of them will be cheating if,

(a) a binomial distribution is assumed

(b) a Poisson distribution is assumed

FYI : If $X \sim B(n, p)$ and n is large and p is small then the Poisson distribution $Y \sim Po(np)$ can be used as an approximation to the binomial distribution