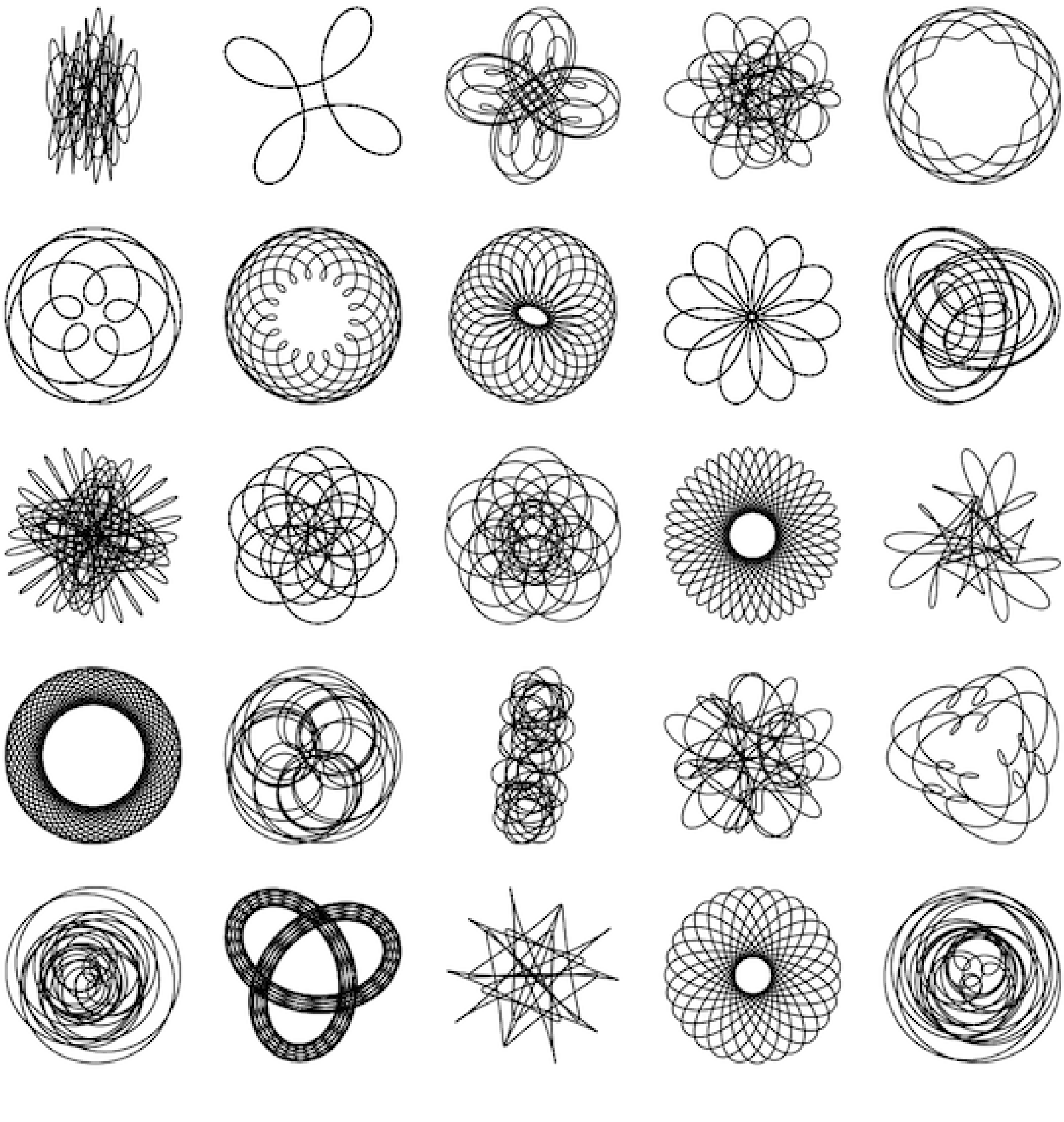
A-Level Pure Mathematics
Year 2

DIFFERENTIATION IV



Parametric Differentiation • Implicit Differentiation

1.1 What are Parametric Equations ?

A key idea in mathematics is the separation of "what is happening in the *x* direction" from "what is happening in the *y* direction". This concept is often first encountered by students in mechanics, where the motion of a projectile is analysed by separating the horizontal component of a projectile's motion from its vertical component.
To elaborate on this example; for a projectile in flight, the horizontal component of the projectile's motion is determined by the single equation,

$$Distance = Speed \times Time$$

whereas the vertical component of a projectile's motion is determined by the following five equations, often referred to collectively as the *suvat* equations,

$v = u + at$	s displacement
$s = vt - \frac{1}{2}at^2$	u initial velocity
$s = ut + \frac{1}{2}at^2$	v final velocity
$s = \left(\frac{v+u}{2}\right)t$	a acceleration (constant)
$v^2 = u^2 + 2as$	t time

The Applied Mathematics topic of Projectiles, has at its heart the fact that the motion of a projectile in the *x* direction is independent of the motion in the *y*.

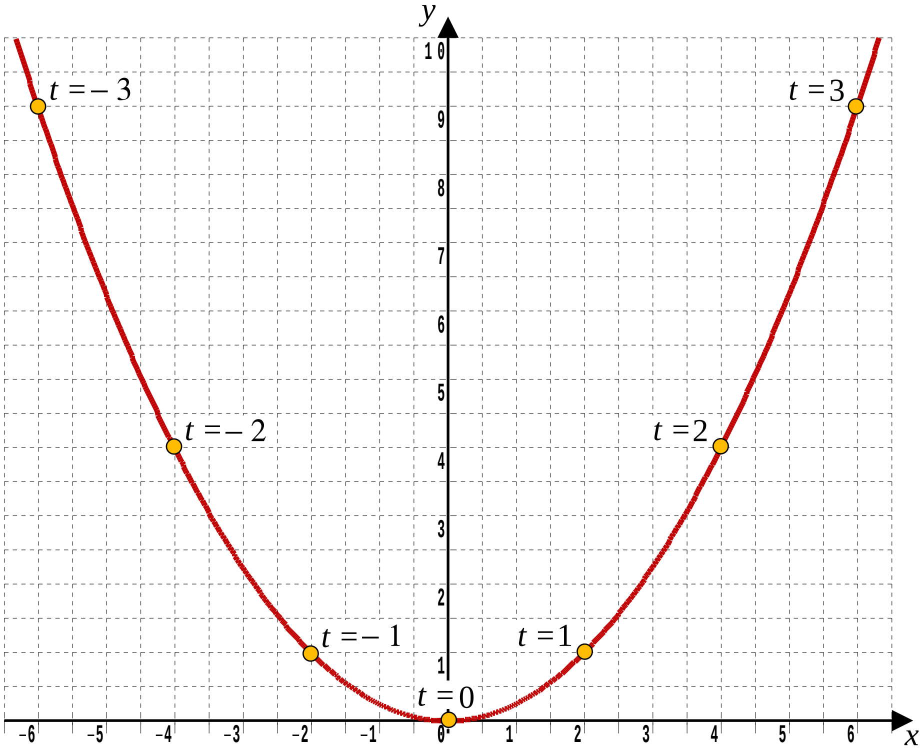
For a simple example to illustrate where, more generally, this separatist line of thought leads, consider the two equations; $x = 2t, \; y = t^2$
Such a pair of equations, one for *x* and a separate one for *y*, each in terms of time *t*, are an example of a pair of parametric equations. The *t* does not have to represent time, and in general it is termed a parameter. As *t* varies through all possible values, all possible points on the path are obtained, thus, in effect, providing another way of plotting a graph.

To plot $x = 2t, \; y = t^2$ work out the points on the path for particular values of t (e.g. when $t = -3, -2, -1, \; 0, \; 1, \; 2, \; 3)$ then join them up, dot-to-dot fashion.

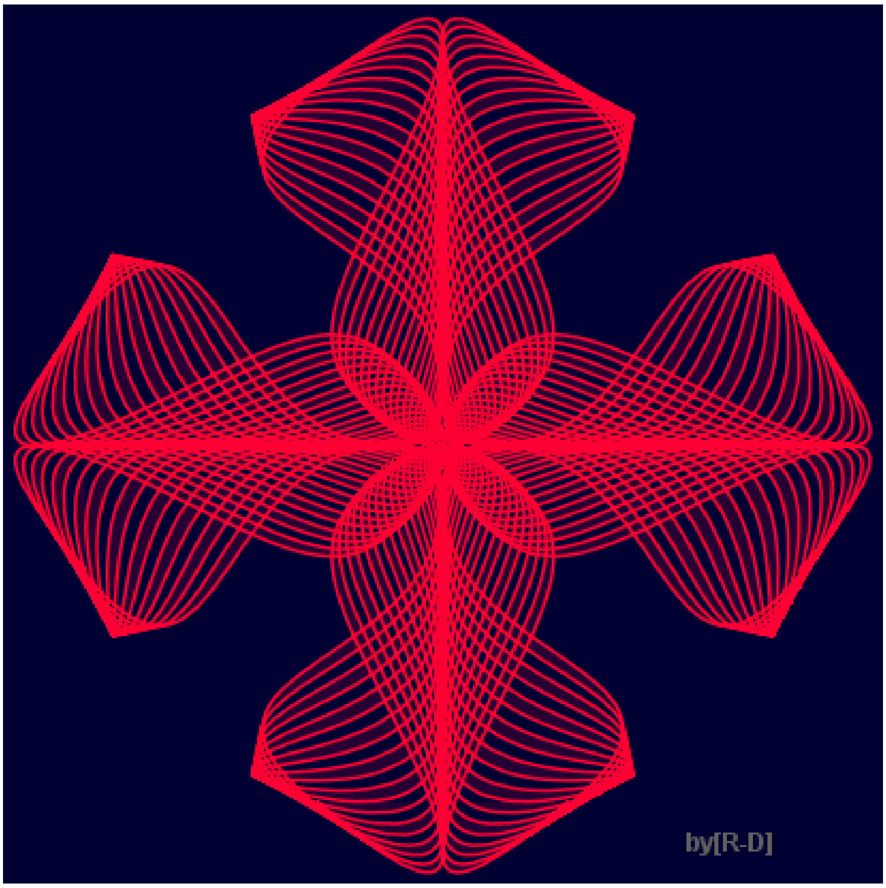
A table keeps the working organised;

t	-3	-2	-1	0	1	2	3
$x = 2t$	-6	-4	-2	0	2	4	6
$y = t^2$	9	4	1	0	1	4	9
(x, y)	$(-6, 9)$	$(-4, 4)$	$(-2, 1)$	$(0, 0)$	$(2, 1)$	$(4, 4)$	$(6, 9)$

And the resulting graph is,



Rather than view this as a static object, it can be considered a path along which a point moves. Isaac Newton saw the point as a particle. It arrives top left at a brisk pace, then slows down as it approaches the origin, and then speeds up again as it leaves, top right.



Some spectacular paths can be generated using parametric equations.

1.2 Exercise

Marks Available : 50

Question 1

(i) A curve is described by the parametric equations

$$x = 3\,sin\,2\theta°$$

$$y = 6\,cos\,\theta° - 3\sqrt{2}\,sin^2\,\theta°$$

Complete the following tables and graph the resulting curve.
Work to 1 decimal place.

θ (in degrees)	0	15	30	45	60	75	90
$x = 3\ sin\ 2\theta°$							
$y = 6\,cos\,\theta° - 3\sqrt{2}\,sin^2\,\theta°$							

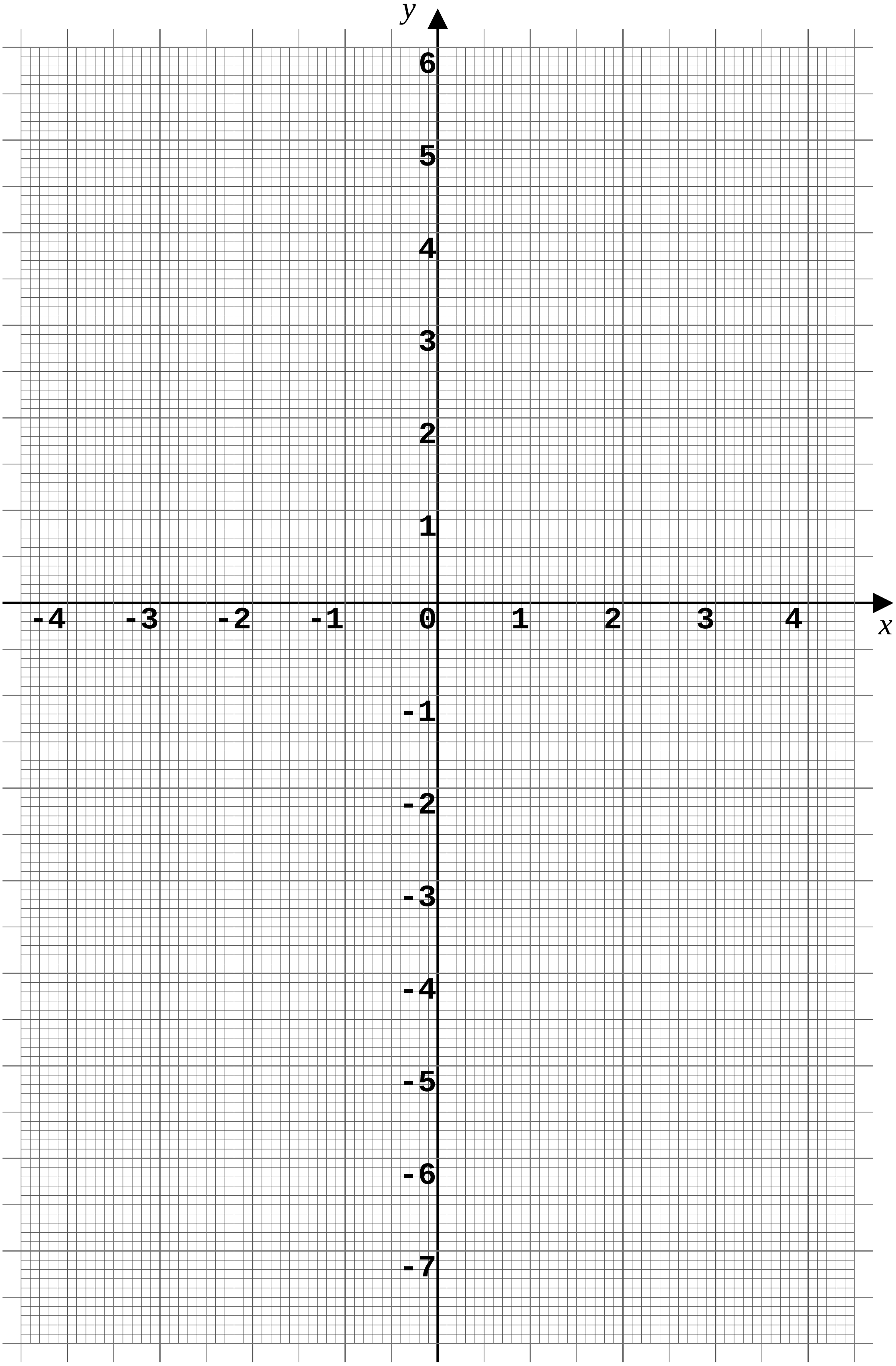
θ (in degrees)	105	120	135	150	165	180
$x = 3\ sin\ 2\theta°$						
$y = 6\,cos\,\theta° - 3\sqrt{2}\,sin^2\,\theta°$						

θ (in degrees)	195	210	225	240	255	270
$x = 3\ sin\ 2\theta°$						
$y = 6\,cos\,\theta° - 3\sqrt{2}\,sin^2\,\theta$						

θ (in degrees)	285	300	315	330	345	360
$x = 3\ sin\ 2\theta°$						
$y = 6\,cos\,\theta° - 3\sqrt{2}\,sin^2\,\theta°$						

[10 marks]

(ii) Plot your part (i) curve.



[10 marks]

Question 2

A curve is described by the parametric equations

$$x = 4t^2$$

$$y = 16t\left(t^2 - 1\right)$$

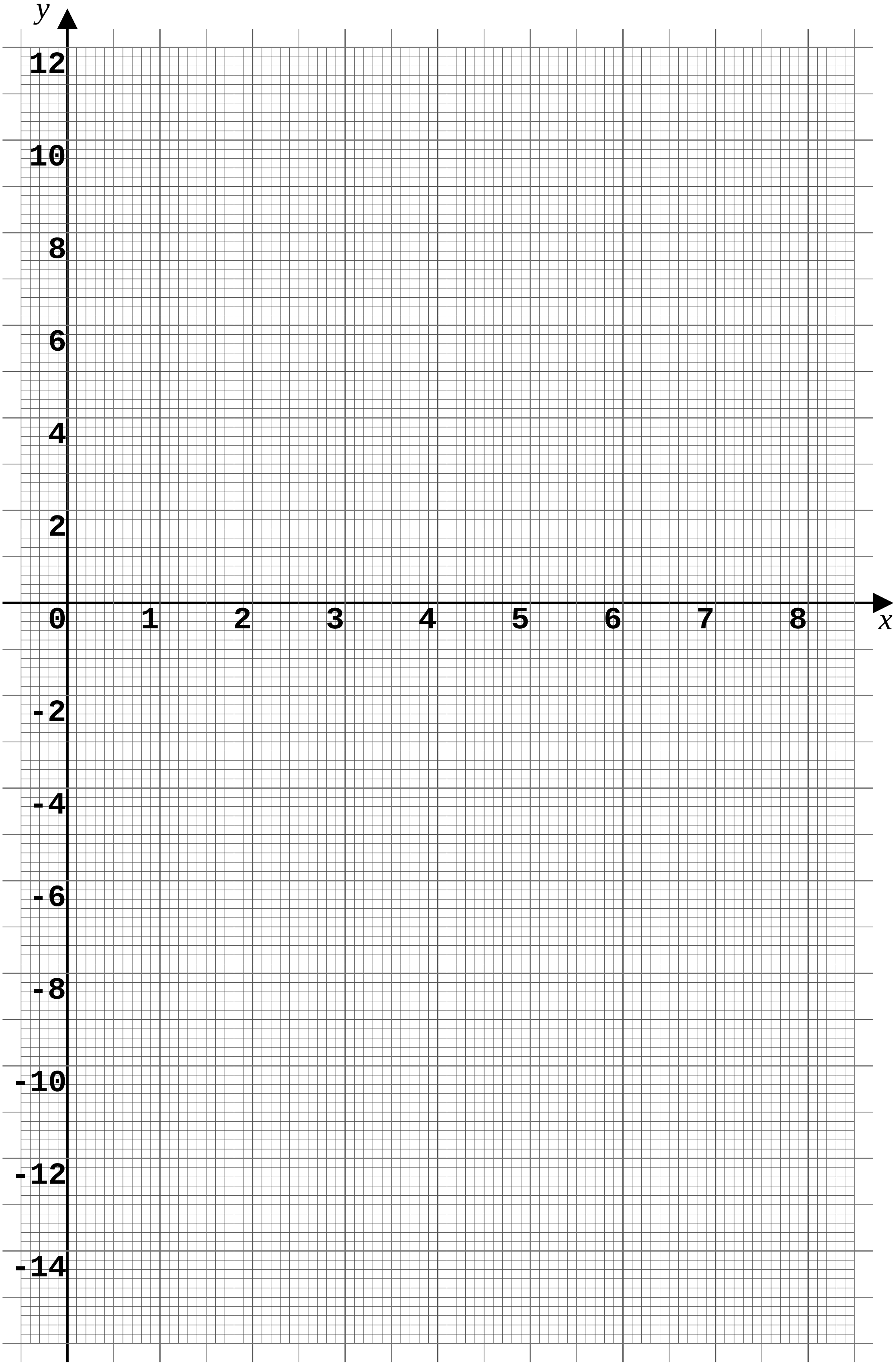
(i) Complete the following table.
Do not round off any table entries.

t	-1.25	-1	-0.75	-0.5	-0.25
$x = 4t^2$					
$y = 16t\left(t^2 - 1\right)$					

t	0	0.25	0.5	0.75	1	1.25
$x = 4t^2$						
$y = 16t\left(t^2 - 1\right)$						

[8 marks]

(ii) Plot your part (i) curve.
You may round off coordinates to 1 decimal place if you wish.



[7 marks]

Question 3

A curve is described by the parametric equations

$$x = 4 \sin \theta$$

$$y = 6 \cos \theta$$

(i) Complete the following table.
Work to 1 decimal place.
Make sure you are working in **RADIANS**

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x = 4 \sin \theta$							
$y = 6 \cos \theta$							

[6 marks]

(ii) Plot your part (i) curve using the graph paper on the following page.

[4 marks]

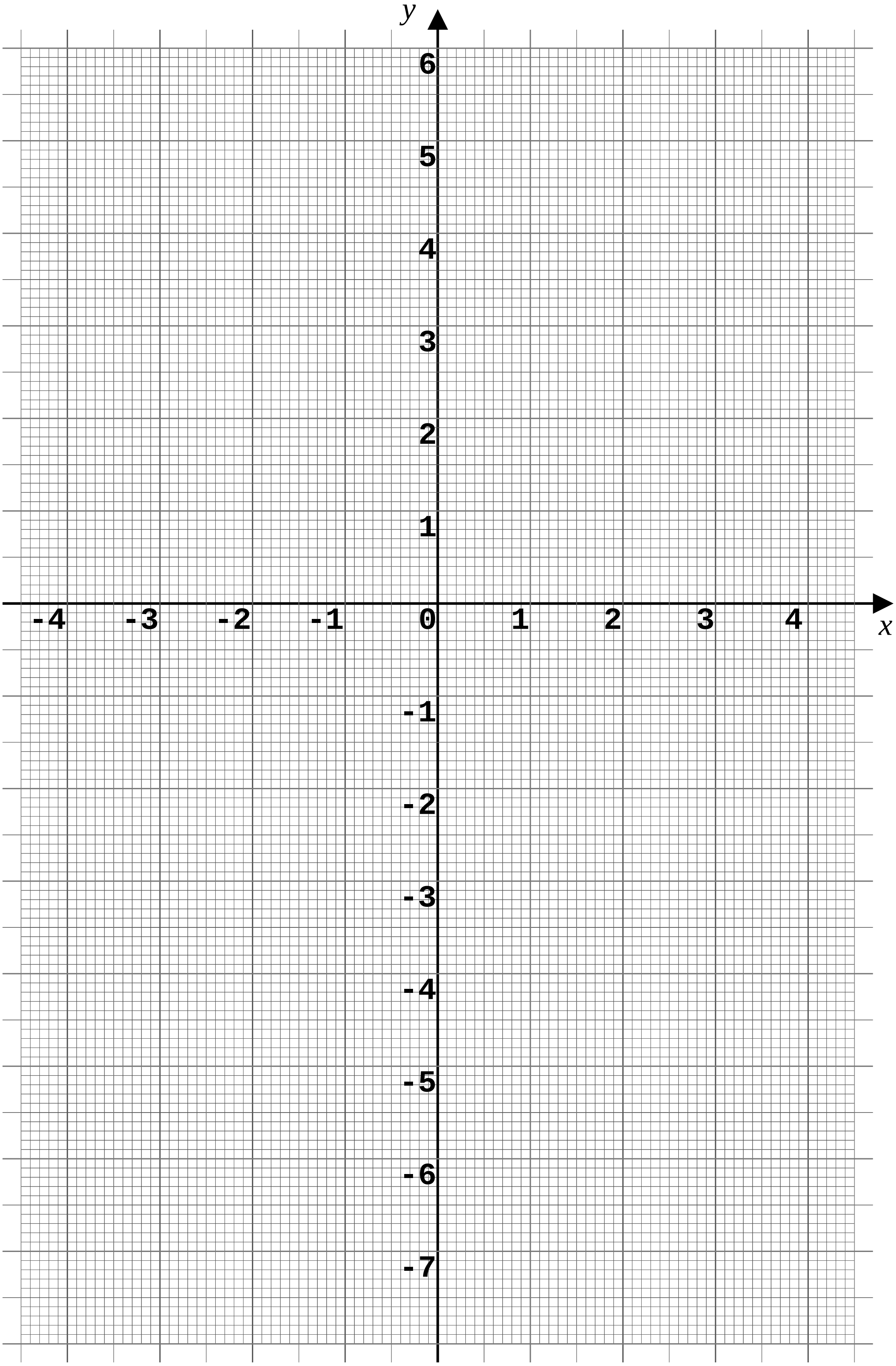
(iii) The table in part (i) was for $0 \leqslant \theta \leqslant \pi$

Draw the additional part of the graph for $\pi < \theta \leqslant 2\pi$

[3 marks]

(iv) Explain why the graph for $2\pi \leqslant \theta \leqslant 4\pi$ would not look any different to that for $0 \leqslant \theta \leqslant 2\pi$

[2 marks]


y
6
5
4
3
2
1
-4
-3
-2
-1
0
1
2
3
4
x
-1
-2
-3
-4
-5
-6
-7