Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

COMPLEX NUMBERS I

C O M P L E X   N U M B E R S   I

Lesson 1

Further A-Level Pure Mathematics : Core 1
Complex Numbers I

1.1 Introduction

Welcome to the start of the Further Mathematics Course. The first topic, Complex Numbers, is part of the Core 1 section, compulsory for all further mathematicians at both A and AS Level.

1.2 What is a complex number ?

When solving a quadratic equation such as,

$$z^2 - 4z + 5 = 0$$

one method is to utilise the method of completing the square.

Here is what a typical solution would look like,

$$z^2 - 4z + 5 = 0$$
$$\left[\, z^2 - 4z \,\right] + 5 = 0$$
$$\left[\, (z - 2)^2 - 4 \,\right] + 5 = 0$$
$$(z - 2)^2 + 1 = 0 \qquad \text{This is completed square form}$$
$$(z - 2)^2 = -1$$
$$z - 2 = \pm\sqrt{-1} \qquad \text{By square rooting both sides}$$
$$z = 2 \pm \sqrt{-1}$$
$$z = 2 \pm \mathrm{i}$$

The first four lines of this solution are familiar territory. Depending upon what you already knows about complex numbers will determine how you feel about the final four lines. Historically, mathematicians would abandon the solution at the point where $\sqrt{-1}$ occurred. However, embracing the strange final result opened a doorway into a wonderful and enlarged world of mathematics; All that was already accepted sat quite comfortably within a new land in which $\mathrm{i} = \sqrt{-1}$ and $\mathrm{i}^2 = -1$. As a subject, complex numbers became mainstream in the 18th century but they had been known “as a curiosity” for around two hundred years before that.

When solving any equation, one method to check that a solution is valid is to substitute the proposed solution back into the original equation. If it is a valid solution then it will make the original equation true. For example, to check that $x = 4$ is a solution to the equation $5x + 3 = 23$, replace the x on the left hand side with 4, and show that it then turns the left hand side into the right hand side. So, by way of introducing some complex number arithmetic, it will now be shown that $2 + \mathrm{i}$ is a valid solution to the equation $z^2 - 4z + 5 = 0$

1.3 Complex Number Arithmetic

Show by means of substitution that $2 + \mathrm{i}$ is a solution to $z^2 - 4z + 5 = 0$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(1).mp4

[3 marks]

1.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Use the method of completing the square to show that the quadratic equation,

$$z^2 - 6z + 13 = 0$$

has solutions,

$$z = 3 \pm 2\mathrm{i}$$

[3 marks]

Question 2

Use the method of completing the square to show that the quadratic equation,

$$z^2 - 4z + 29 = 0$$

has solutions

$$z = 2 \pm 5\mathrm{i}$$

[3 marks]

Question 3

Use the method of completing the square to show that the quadratic equation,

$$z^2 + 8z + 21 = 0$$

has solutions

$$z = -4 \pm \sqrt{5}\,\mathrm{i}$$

[3 marks]

Question 4

The quadratic equation $az^2 + bz + c = 0$ has solutions given by the formula,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show how this formula could be used to solve the quadratic equation,

$$z^2 + 5z + 25 = 0$$

Present your answer in a simplified, elegant form.

[3 marks]

Question 5

Show by means of substitution that $z = 3 + \mathrm{i}$ is a solution of the equation,

$$z^2 - 6z + 10 = 0$$

[3 marks]

Question 6

$$f(z) = z^2 - 2z + 17$$

Show by means of substitution that $z = 1 - 4\mathrm{i}$ is a solution to $f(z) = 0$

[3 marks]

Question 7

(i) Show by substitution that $z = 1 + \mathrm{i}$ is a solution of the equation,

$$z^4 + 4 = 0$$

[4 marks]

(ii) Guess another solution to the equation $z^4 + 4 = 0$ and then verify that your guess is good.

[4 marks]

Question 8

Prove that $(a + b\mathrm{i})(a - b\mathrm{i})$ is a real number for any real numbers a and b

[3 marks]

Question 9

(i) Pick any two positive real integers a and b such that $a > b$.

[1 mark]

(ii) Calculate

$$(a + b\mathrm{i})^2$$

writing your answer in the form $u + v\mathrm{i}$ where u and v are real integers.

[2 marks]

(iii) Calculate $\sqrt{u^2 + v^2}$

[1 mark]

(iv) Demonstrate that, not only is $\sqrt{u^2 + v^2}$ an integer, but that u, v and $\sqrt{u^2 + v^2}$ are a Pythagorean integer triple.

[3 marks]

Isn't that amazing ?

[0 marks]

Question 10

Use the method of completing the square to show that the quadratic equation,

$$7z^2 - 3z + 3 = 0$$

has solutions

$$z = \frac{3}{14} \pm \frac{5\sqrt{3}}{14}\,\mathrm{i}$$

[4 marks]