## 4.1 To Simplify, Complexify

Mathematicians' ongoing fascination with Polynomials stems from the fact that they are one of the most simple class of function. Many other more complicated functions can be approximated with a polynomial.

For example, using the Binomial Theorem,

$$\sqrt{4 + x} = 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \ldots \qquad x \in \mathbb{R}, \qquad |x| < 4$$

Polynomials over the reals are, of course, easy to differentiate and integrate. When over the real numbers they have a structure that is very like that of the prime numbers and, in particular, they have the property of unique factorisation. However, one unsatisfactory consequence of restricting polynomials to the real numbers is that the number of roots for a polynomial of any given degree varies. For example, consider the generalised polynomial of degree two, a quadratic;

$$a x^2 + bx + c = 0$$

There are,

- Two roots if the discriminant is positive
- One root (sometimes called a repeated root) is the discriminant is zero
- No roots at all if the discriminant is negative

In moving to complex numbers, the desirable property of unique factorisation is retained except for the trivial case of a polynomial of degree zero. One of the many benefits of working with polynomials over the complex numbers is that the number of roots for a polynomial of any given degree no longer varies. This is of such importance that it has the grand title, “The fundamental theorem of algebra”.

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**The Fundamental Theorem of Algebra**

Any polynomial in $z$ of degree $n$ has exactly $n$ roots, where some or all of the roots may be complex numbers.

The polynomial can even have complex coefficients.

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An oft repeated mathematical quote is,

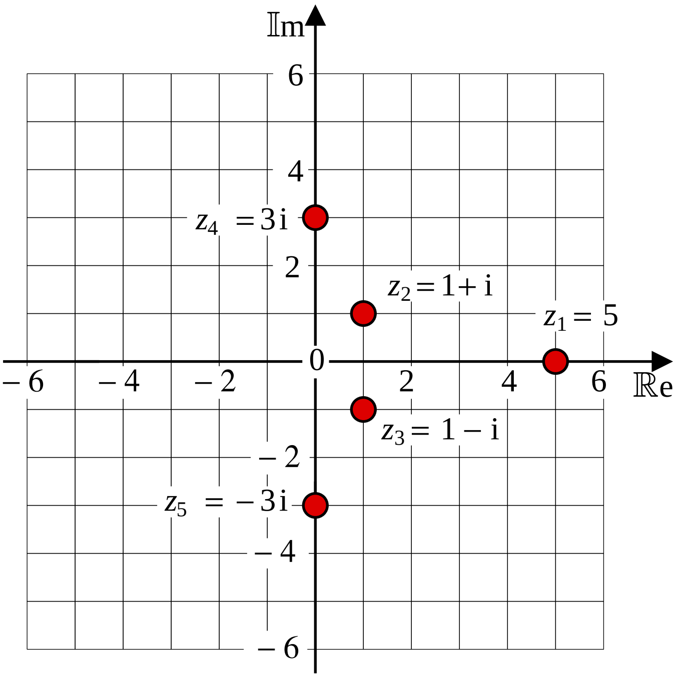
“to simplify, complexify”

which is saying that for many mathematical problems the easy way to tackle them is to consider them, not just over the real numbers, but in the world of the complex numbers. The “problems” referred to are not just in mathematics. Complex numbers have traditionally been used in physics and engineering. In electrical engineering, for example, circuits involving resistors, capacitors and inductors are far easier and more elegantly modelled using complex numbers. More recently, complex numbers have become a part of Biology with the realisation that nature can be effectively modelled using Fractal Geometry which has, at its core (you guessed it) Complex Numbers !

### 4.2 A Polynomial From Its Roots

In Lesson 3 the problem of finding the roots of a polynomial was tackled, often using a few clues, such as being given one root and asked to find the others. The factor theorem was useful along with the knowledge that, if the polynomial had only real coefficients, any complex roots came as a conjugate pair. How would this work in reverse ? In other words, given the roots of a polynomial, how might the polynomial they came from be derived ?

The roots of a mystery polynomial are shown below on an Argand diagram.
Find a polynomial that they could be the root of.




Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(4).mp4

| | $z^3$ | $-5z^2$ | $+9z$ | $-45$ |
| --- | --- | --- | --- | --- |
| $z^2$ | | | | |
| $-2z$ | | | | |
| $+2$ | | | | |

Multiplication Grid the for Teaching Video

**[ 5 marks ]**

The final statement in the video is not precise : It should be that "This polynomial is the only polynomial with highest order term $z^5$ (note the coefficient of 1) that will have $z_1$, $z_2$, $z_3$, $z_4$ and $z_5$ as its only roots".

Question 1 (iii) in Exercise 4.3, will emphasise this subtlety.

**4.3 Exercise**

*Any solution based entirely on graphical*
*or numerical methods is not acceptable*
Marks Available : 40

**Question 1**

A quadratic has roots,

$$z_1 = 7 + 5\mathrm{i} \quad \text{and} \quad z_2 = 7 - 5\mathrm{i}$$

**( i )** Write down the two factors of the quadratic.

**[ 1 mark ]**

**( ii )** Show how the factors may be multiplied together to give the quadratic,

$$z^2 - 14z + 74$$

**[ 2 marks ]**

**( iii )** Explain why

$$2z^2 - 28z + 148 = 0$$

would have the same roots.

**[ 2 marks ]**

**Question 2**

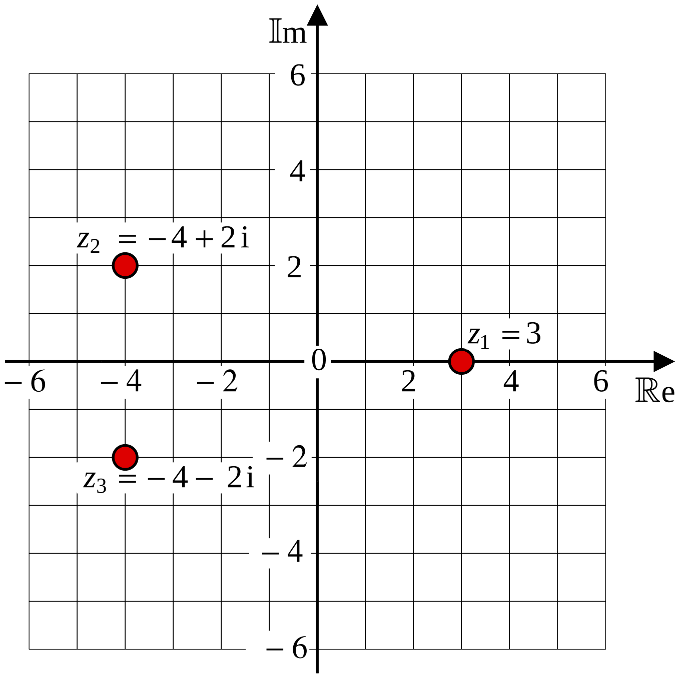
A polynomial, $p(z)$ has the four roots,

$$z_1 = 1 \quad z_2 = \mathrm{i} \quad z_3 = -1 \quad \text{and} \quad z_4 = -\mathrm{i}$$

Find a quartic polynomial that will have $z_1$, $z_2$, $z_3$ and $z_4$ as its only roots.

**[ 4 marks ]**

**Question 3**




**( i )** Which two roots form a conjugate pair ?

**[ 1 mark ]**

**( ii )** What is the mathematical theorem called that tells you that $p(z)$ will be a cubic ?

**[ 1 mark ]**

**( iii )** Explain why $p(z)$ will have real coefficients.

**[ 1 mark ]**

**( iv )** Find a cubic that will have $z_1$, $z_2$ and $z_3$ as its only roots.

**[ 4 marks ]**

**Question 4**

*Further A-Level Examination Question from June 2014, FP1, Q3*

Given that 2 and $1 - 5\mathrm{i}$ are roots of the equation

$$x^3 + px^2 + 30x + q = 0 \qquad p, q \in \mathbb{R}$$

**( a )** write down the third root of the equation

**[ 1 mark ]**

**( b )** Find the value of $p$ and the value of $q$

**[ 5 marks ]**

**( c )** Show the three roots of this equation on a single Argand diagram

**[ 2 marks ]**

**Question 5**

*Further A-Level Examination Question from May 2016, FP1, Q4*

$$z = \frac{4}{1 + \mathrm{i}}$$

Find in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$

**( a )** $z$

**[ 2 marks ]**

**( b )** $z^2$

**[ 2 marks ]**

Given that $z$ is a complex root of the quadratic equation,

$$x^2 + px + q = 0$$

where $p$ and $q$ are real integers,

**( c )** find the value of $p$ and the value of $q$

**[ 3 marks ]**

**Question 6**

*Further A-Level Examination Question from June 2019, Core 1, Q1*

$$f(z) = z^4 + a z^3 + b z^2 + cz + d$$

where $a$, $b$, $c$ and $d$ are real constants.

Given that $-1 + 2\mathrm{i}$ and $3 - \mathrm{i}$ are two roots of the equation $f(z) = 0$

**( a )** show all the roots of $f(z) = 0$ on a single Argand diagram

**[ 4 marks ]**

**( b )** find the values of $a$, $b$, $c$, and $d$

**[ 5 marks ]**