2.1 Root Relations I

Given a function, a standard mathematical technique is to move it to a place where it is more easily studied, before then moving the results of the studies back to the original location. With polynomials, it's often easiest to move the polynomial by moving it's roots rather than the polynomial itself. All the more so as the actual values of the roots need not be found. The “move” need not be a simple translation and, indeed, all sorts of interesting other transformations may come into play.

2.2 Cardano's Use of Root Relations

In the last question of last lesson's exercise it was shown that this process was used by Cardano with his formula for solving cubic equations.

Faced with a general cubic,

$$a x^3 + b x^2 + cx + d = 0$$

he would move it by replacing x with $t - \dfrac{b}{3a}$

This always resulted in what is termed a “depressed cubic”, one of the form,

$$t^3 + pt + q = 0$$

Cardano knew how to find the roots of this, and by moving those roots back he then had the roots of the original cubic.

Gerolamo Cardano (1501-1576)
An Italian polymath who was one of the most influential mathematicians of the Renaissance.
Today, he is most remembered for his achievements in algebra.
He made the first systematic use of negative numbers in Europe, published with attribution the solutions of other mathematicians for the cubic and quartic equations
and acknowledged the existence of imaginary numbers.

2.3 Example

Let the roots of the quadratic equation $2x^2 + 7x + 1 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$2\alpha + 1 \quad \text{and} \quad 2\beta + 1$$

Teaching Video : http://www.NumberWonder.co.uk/v9093/2.mp4

After watching the teaching video write out your own version of the solution.

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[5 marks]

2.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

The roots of the quadratic equation $3x^2 - 4x + 6 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$3\alpha + 1 \quad \text{and} \quad 3\beta + 1$$

[5 marks]

Question 2

The roots of the quadratic equation $3x^2 + 6x + 1 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$1 - \alpha \quad \text{and} \quad 1 - \beta$$

[5 marks]

Question 3

The roots of the quadratic equation $2x^2 + 3x + 7 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$\alpha^2 + 1 \quad \text{and} \quad \beta^2 + 1$$

[6 marks]

Question 4

The roots of the quadratic equation $5x^2 + 15x + 8 = 0$ are α and β.

Without solving the equation, find an all integer coefficient polynomial with roots

$$1 + \frac{1}{\alpha} \quad \text{and} \quad 1 + \frac{1}{\beta}$$

[8 marks]

Question 5

Further A-Level Examination Question from January 2018, IAL, F1, Q4

The quadratic equation $3x^2 + 2x + 5 = 0$ has roots α and β

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$

[2 marks]

(b) show that $\alpha^3 + \beta^3 = \dfrac{82}{27}$

[2 marks]

(c) find a quadratic equation with roots $\left(\alpha + \dfrac{\alpha}{\beta^2} \right)$ and $\left(\beta + \dfrac{\beta}{\alpha^2} \right)$

Give the answer in the form $px^2 + qx + r = 0$, with $p, q, r \in \mathbb{Z}$

[4 marks]

Question 6

This question is about using Cardano's method to find the roots of the cubic,

$$x^3 - 66x + 340 = 0$$

It's a little more challenging than the question at the end of previous exercise although the cubic is already in depressed form so there is no need to change the variable. Here is a recap of the theory;

Roots of a Cubic

Given a depressed cubic of the form

$$t^3 + pt + q = 0 \qquad\qquad p,\ q \in \mathbb{C}$$

where p and q are not both zero, and $4p^3 + 27q^2 \neq 0$, calculate,

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A root of the cubic is then given by,

$$\alpha = C - \frac{p}{3C}$$

Polynomial division can then be used to find the remaining roots.

To help with a tricky step, there is a part (i) which is useful when tackling the main problem.

(i) Expand the brackets,

$$\left(\sqrt{3} - 5\right)^3$$

giving your answer in the form $a\sqrt{3} + b$ for integer a and b

[2 marks]

(ii) Determine, using the method of Cardano, the real root of the cubic,

$$x^3 - 66x + 340 = 0$$

[4 marks]

(iii) Using polynomial division with your part (ii) answer, find all three roots, two of which are a complex conjugate pair, of the depressed cubic,

$$x^3 - 66x + 340 = 0$$

[2 marks]