5.1 Root Algebra Galore

In questions about the roots of quadratic and cubic equations, the same pieces of algebra repeatedly occur and most mathematicians end up with their own personal catalogue of algebraic relationships that they have encountered and found useful. Here is presented a beginners catalogue, based on the work of this topic. In examinations, you may use these algebraic relationships without proof.

Reciprocals

Quadratic : $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$$

Cubic : $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

Quartic : $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta}$$

Sums of Squares

Quadratic : $$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

Cubic : $$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

Quartic : $$\alpha^2 + \beta^2 + \gamma^2 + \delta^2$$
$$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta))$$

Sums of Cubes

Quadratic : $$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

Cubic : $$\alpha^3 + \beta^3 + \gamma^3$$
$$= (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

Sum of Quartics

Quadratic : $$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$$

5.2 The Roots of a Quartic

The Roots of a Quartic

If α, β, γ and δ are roots of

$$a x^4 + b x^3 + c x^2 + dx + e = 0 \qquad a, b, c, d, e \in \mathbb{C}$$

then, $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$ $\qquad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta) = \frac{c}{a}$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a} \qquad \alpha\beta\gamma\delta = \frac{e}{a}$$

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

α, β, γ and δ are the roots of the quartic, $x^4 + 3x^3 + 2x^2 - x + 4 = 0$
Without solving the equation, find the values of,

(i) $\alpha + \beta + \gamma + \delta$ **(ii)** $\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta)$

[1, 1 mark]

(iii) $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta$ **(iv)** $\alpha\beta\gamma\delta$

[1, 1 mark]

(v) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

[2 marks]

(vi) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

[2 marks]

Question 2

The roots of a quartic equation $a x^4 + b x^3 + c x^2 + dx + e = 0$ are,

$$\alpha = -3, \ \beta = -1, \ \gamma = 2 \text{ and } \delta = 5$$

Find integer values for a, b, c, d and e

[4 marks]

Question 3

The quartic equation $2x^4 - 34x^3 + 202x^2 + dx + e = 0$ has roots,

$$\alpha, \quad \alpha + 1, \quad 2\alpha + 1 \quad \text{and} \quad 3\alpha + 1$$

(i) Find the value of α

[2 marks]

(ii) Find the values of d and e

[4 marks]

Question 4

(a) Prove the "Sum of Quartics" formula for a quadratic equation with roots α and β, that,

$$\alpha^4 + \beta^4 = \left(\alpha^2 + \beta^2 - \sqrt{2}\,\alpha\beta\right)\left(\alpha^2 + \beta^2 + \sqrt{2}\,\alpha\beta\right)$$

[2 marks]

(b) For the quadratic equation $7x^2 + 13x - 8 = 0$ with roots α and β, find, without solving the equation, the value of,

(i) $\alpha^2 + \beta^2$

[2 marks]

(ii) $\alpha^4 + \beta^4$

[4 marks]

Question 5

Further A-Level Examination Question from January 2015, Q5 (Edexcel)

The quadratic equation $4x^2 + 3x + 1 = 0$ has roots α and β.

(a) Write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$

[2 marks]

(b) Find the value of $\left(\alpha^2 + \beta^2\right)$

[2 marks]

(c) Find a quadratic equation with roots,

$$(4\alpha - \beta) \quad \text{and} \quad (4\beta - \alpha)$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

[4 marks]

Question 6

The cubic equation $x^3 - 14x^2 + 56x - 64 = 0$ has roots α, $k\alpha$ and $k^2\alpha$ for some real constant k.
Find the two possible values of α and the corresponding two values of k.

[6 marks]