

Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

P R O O F

B Y

I N D U C T I O N



PROOF BY INDUCTION

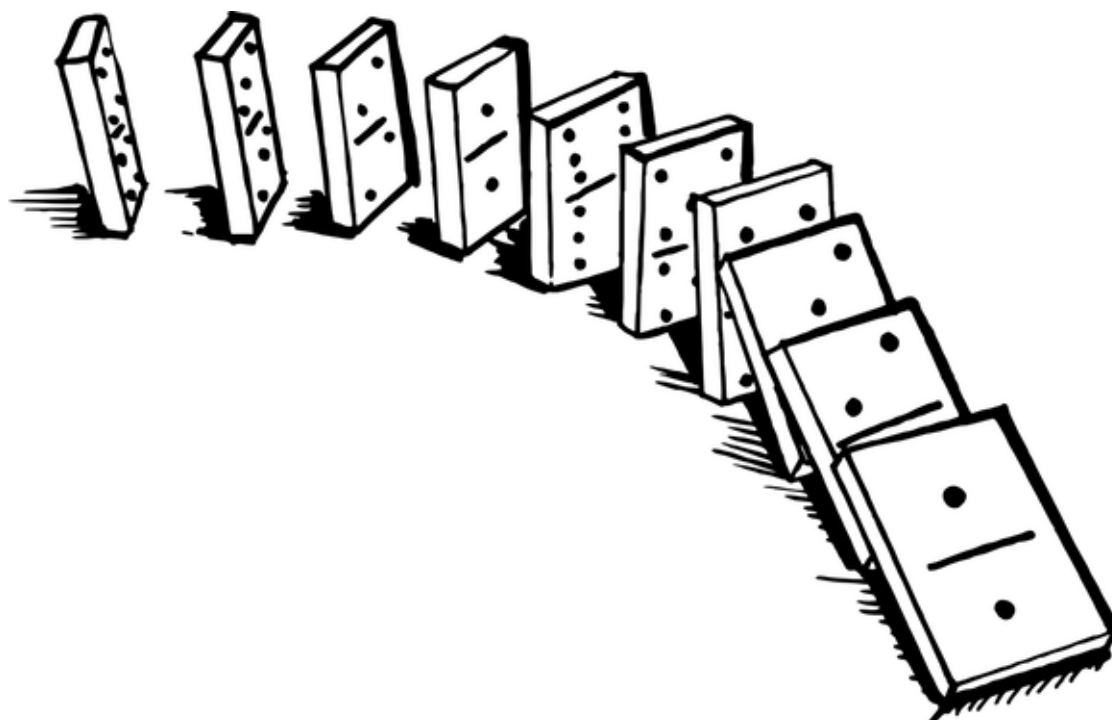
Lesson 1

Further A-Level Pure Mathematics : Core 1

Proof by Induction

1.1 The Domino Effect

Proof by induction is often likened to an imaginary line of infinite extent of dominoes. To prove that all of the dominoes will fall over one must prove, firstly, that the first domino in the line will fall over and, secondly, that if any domino in the line falls over it will cause the domino after it to also fall over.



1.2 Example

Suppose that numbers in the sequence $E_n = 3^{2n} + 11$ are to be investigated.

n	1	2	3	4	5	6
E_n	20	92	740	6572	59060	531452
Prime Factorisation	$2^2 \cdot 5$	$2^2 \cdot 23$	$2^2 \cdot 5 \cdot 37$	$2^2 \cdot 31 \cdot 53$	$2^2 \cdot 5 \cdot 2953$	$2^2 \cdot 132863$

From looking at the data in the table, the conjecture could be made that all numbers in this sequence are divisible by 4 which is the most striking feature for the first six terms in the sequence.

However, there is uncertainty as to whether or not the property of being divisible by 4 will continue to hold for all subsequent terms.

To resolve the matter a proof is needed one way or the other.

Statement

For all positive integers n , $3^{2n} + 11$ is divisible by 4

Proof by Induction

Let $f(n) = 3^{2n} + 11$, $n \in \mathbb{Z}^+$

When $n = 1$, $f(1) = 3^{2 \times 1} + 11$

$$= 20 \quad \text{which is divisible by 4}$$

Assume that when $n = k$, $f(k) = 3^{2k} + 11$ is divisible by 4
in which case...

$$\begin{aligned} f(k+1) &= 3^{2(k+1)} + 11 \\ &= 3^{2k} \times 3^2 + 11 \\ &= 9 \times 3^{2k} + 11 \\ &= 8(3^{2k}) + 3^{2k} + 11 \\ &= 8(3^{2k}) + f(k) \\ &= 4(2(3^{2k})) + f(k) \\ \therefore f(k+1) &\text{ is divisible by 4} \end{aligned}$$

Therefore, if $f(n)$ is divisible by 4 when $n = k$,

then $f(n)$ is also divisible by 4 when $n = k + 1$

As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 4 for all $n \in \mathbb{Z}^+$
by mathematical induction. □

[6 marks]

The Teaching Video will talk you through the above proof.

Teaching Video : <http://www.NumberWonder.co.uk/v9096/1.mp4>



1.3 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 40

Question 1

Numbers in the sequence $D_n = n^3 + 2n$ are to be investigated.

n	1	2	3	4	5	6
D_n	3	12	33	72	135	228
Prime Factorisation	3	$2^2 \cdot 3$	$3 \cdot 11$	$2^3 \cdot 3^2$	$3^3 \cdot 5$	$2^2 \cdot 3 \cdot 19$

From looking at the data it would seem that for all positive integers, n , the terms in the sequence D_n are divisible by 3.

Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 2

Numbers in the sequence $P_n = 3^{2n} - 1$ are to be investigated.

n	1	2	3	4	5	6
P_n	8	80	728	6560	59048	531440
Prime Factorisation	2^3	$2^4 \cdot 5$	$2^3 \cdot 7 \cdot 13$	$2^5 \cdot 5 \cdot 41$	$2^3 \cdot 11^2 \cdot 61$	$2^4 \cdot 5 \cdot 7 \cdot 13 \cdot 73$

From looking at the data it would seem that for all positive integers, n , the terms in the sequence P_n are divisible by 8

Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 3

- (i) Explain why the product of any two consecutive integers will divide by 2.

[1 mark]

- (ii) Explain why $51k(k + 1)$ is divisible by 6 for any integer k

[2 marks]

- (iii) Numbers in the sequence $Z_n = 17n^3 + 103n$ are to be investigated.

n	1	2	3	4	5	6
Z_n	120	342	768	1500	2640	4290
Prime Factorisation	$2^3 \cdot 3 \cdot 5$	$2 \cdot 3^2 \cdot 19$	$2^8 \cdot 3$	$2^2 \cdot 3 \cdot 5^3$	$2^4 \cdot 3 \cdot 3 \cdot 11$	$2 \cdot 3 \cdot 5 \cdot 11 \cdot 13$

From looking at the data it would seem that for all positive integers, n , the terms in the sequence Z_n are divisible by 6.

Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 4

Numbers in the sequence $f(n) = 13^n - 6^n$ are to be investigated.

n	1	2	3	4	5	6
$f(n)$	7	133	1981	27265	363517	4780153
Prime Factorisation	7	7.19	7.283	5.7.19.41	7.11.4721	7.19.127.283

From looking at the data it would seem that for all positive integers, n , the terms in the sequence A_n are divisible by 7

(i) Show that $f(k + 1) = 6f(k) + 7(13^k)$

[3 marks]

(ii) Hence, or otherwise, prove by induction that for all positive integers, n , $f(n)$ is divisible by 7

[4 marks]

Question 5

Further A-Level Examination Question from June 2019, Core 1, Q6 (Edexcel)

Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

[6 marks]

Question 6

Further A-Level Examination Question from January 2018, IAL, F1, Q8 (ii) (Edexcel)

Prove by induction, for all positive integers n ,

$$f(n) = 3^{2n+3} + 40n - 27 \text{ is divisible by } 64$$

[6 marks]

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In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”

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Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk