3.1 Matrices and Induction

This lesson focuses on matrix based statements that can be proved using induction.
The overall structure of the proofs are the same.

3.2 Example

Prove by induction the following statement,

Statement

If $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$ for all positive integers, n

Proof by Induction

When $n = 1$, $\mathbf{A}^1 = \begin{pmatrix} 1 & 0 \\ 5 \times 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ which matches that given for **A**

Assume that when $n = k$, $\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix}$ in which case...

$$
\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\
&= \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 5k + 5 & 1 \end{pmatrix} \qquad \text{* Must show unsimplified matrix *} \\
&= \begin{pmatrix} 1 & 0 \\ 5(k + 1) & 1 \end{pmatrix}
\end{aligned}
$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$
As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction ☐

[6 marks]

The Teaching Video will talk you through the above proof.

Teaching Video : http://www.NumberWonder.co.uk/v9096/3.mp4

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Prove by induction the following statement,

Statement

If $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$ for all positive integers, n

[6 marks]

Question 2
Prove by induction the following statement,

Statement
If $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} 2n + 1 & -4n \\ n & -2n + 1 \end{pmatrix}$ for all positive integers, n

[6 marks]

Question 3

Prove by induction that for all positive integers n,

$$\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{pmatrix}$$

Hint : One way to do this question would be to let $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ and observe the question is claiming that the following statement is true,

Statement

If $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$ then $\mathbf{A}^n = \begin{pmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

[6 marks]

Question 4

Further A-Level Examination Question from May 2018, Core 1, Q8 (Edexcel)

Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

[6 marks]

Question 5

Further A-Level Examination Question from 2018, Mock Core 2, Q6 (Edexcel)

Prove by induction, that for all positive integers n,

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}\left(n^2 + 3n\right) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

[6 marks]