

## Lesson 3

### Further A-Level Pure Mathematics : Core 1

#### Proof by Induction

#### 3.1 Matrices and Induction

This lesson focuses on matrix based statements that can be proved using induction.  
The overall structure of the proofs are the same.

#### 3.2 Example

Prove by induction the following statement,

##### Statement

If  $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$  then  $\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$  for all positive integers,  $n$

##### Proof by Induction

When  $n = 1$ ,  $\mathbf{A}^1 = \begin{pmatrix} 1 & 0 \\ 5 \times 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$  which matches that given for  $\mathbf{A}$

Assume that when  $n = k$ ,  $\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix}$  in which case...

$$\begin{aligned}\mathbf{A}^{k+1} &= \mathbf{A}^k \mathbf{A} \\ &= \begin{pmatrix} 1 & 0 \\ 5k & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 5k + 5 & 1 \end{pmatrix} \quad \text{* Must show unsimplified matrix *} \\ &= \begin{pmatrix} 1 & 0 \\ 5(k + 1) & 1 \end{pmatrix}\end{aligned}$$

Therefore, if the result is true for  $n = k$ , then it is true for  $n = k + 1$

As the result has been shown to be true for  $n = 1$ , the conclusion is that  
it is true for all positive integers by mathematical induction  $\square$

[ 6 marks ]

The Teaching Video will talk you through the above proof.

Teaching Video : <http://www.NumberWonder.co.uk/v9096/3.mp4>



### 3.3 Exercise

*Any solution based entirely on graphical  
or numerical methods is not acceptable*

Marks Available : 30

#### Question 1

Prove by induction the following statement,

#### Statement

If  $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$  then  $\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$  for all positive integers,  $n$

[ 6 marks ]

**Question 2**

Prove by induction the following statement,

**Statement**

If  $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$  then  $\mathbf{A}^n = \begin{pmatrix} 2n + 1 & -4n \\ n & -2n + 1 \end{pmatrix}$  for all positive integers,  $n$

[ 6 marks ]

**Question 3**

Prove by induction that for all positive integers  $n$ ,

$$\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} -3n + 1 & 9n \\ -n & 3n + 1 \end{pmatrix}$$

**Hint :** One way to do this question would be to let  $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$  and observe the question is claiming that the following statement is true,

**Statement**

If  $\mathbf{A} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$  then  $\mathbf{A}^n = \begin{pmatrix} -3n + 1 & 9n \\ -n & 3n + 1 \end{pmatrix}$  for all  $n \in \mathbb{Z}^+$

[ 6 marks ]

**Question 4**

*Further A-Level Examination Question from May 2018, Core 1, Q8 (Edexcel)*

Prove by induction that for  $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n + 1 & -8n \\ 2n & 1 - 4n \end{pmatrix}$$

[ 6 marks ]

**Question 5**

*Further A-Level Examination Question from 2018, Mock Core 2, Q6 (Edexcel)*

Prove by induction, that for all positive integers  $n$ ,

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

[ 6 marks ]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School

It may be freely duplicated and distributed, unaltered, for non-profit educational use

In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”

© 2022 Number Wonder

Teachers may obtain detailed worked solutions to the exercises by email from [mhh@shrewsbury.org.uk](mailto:mhh@shrewsbury.org.uk)