4.1 Summation Formulae and Induction

Here is a number pattern,

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}$$

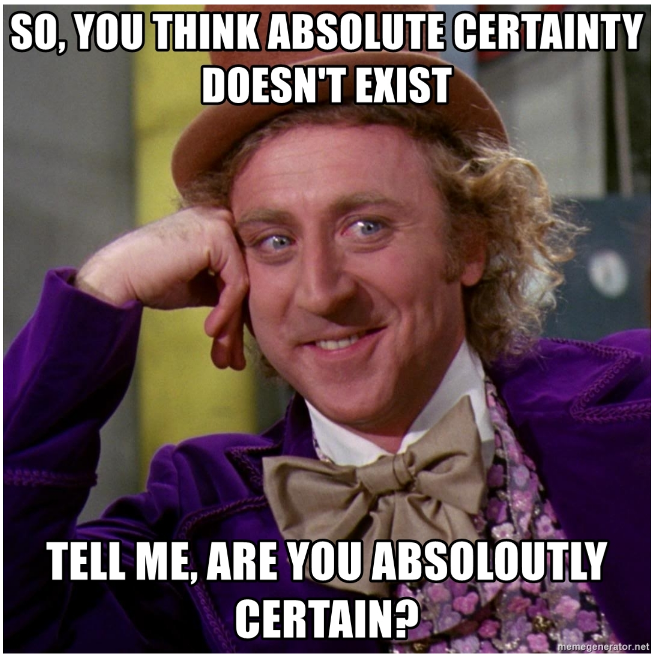
Here is a "reveal" of how the next line of this pattern would be constructed

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6} + \frac{1}{6 \times 7} = \frac{6}{7}$$

The pattern suggests the relationship,

$$\sum_{1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1} \quad \text{for} \quad n \geqslant 1$$

Further exploration may well convince most sceptics that this relationship will continue to hold an the number of terms continues to be increased but, as always, what is really sought is a proof such that the relationship is established with absolute certainty.



4.2 Example

Prove by induction the following statement,

Statement

$$\sum_{1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1} \quad \text{for all positive integers, } n$$

Proof by Induction

When $n = 1$, LHS $= \sum_{1}^{1} \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2}$

RHS $= \frac{1}{1+1} = \frac{1}{2}$

As LHS = RHS the summation formula is true for $n = 1$

Assume that when $n = k$, $\sum_{1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

With $n = k + 1$ the summation formula becomes,

$$\sum_{1}^{k+1} \frac{1}{r(r+1)} = \sum_{1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+1+1)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} \times \frac{(k+2)}{(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{(k+1)}{(k+2)}$$

$$= \frac{(k+1)}{(k+1)+1}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

[6 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9096/4.mp4

<== The Teaching Video will talk through the proof

4.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Consider the following first four lines of a number pattern,

$$\frac{1}{3} = \frac{1}{3}$$

$$\frac{1}{3} + \frac{1}{15} = \frac{2}{5}$$

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} = \frac{3}{7}$$

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} = \frac{4}{9}$$

The left hand side is being constructed from the reciprocals of product pairs of consecutive odd numbers.

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2r - 1)(2r + 1)}$$

The left hand side is conjectured to sum to $\frac{n}{2n + 1}$

(i) Write down the fifth line of the number pattern and verify that it too has the sum suggested.

[2 marks]

(ii) Assuming that the sum of the series is indeed $\frac{n}{2n + 1}$ what is the sum of the series as the number of terms tends towards infinity ?

That is, what is,

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots$$

[2 mark]

(iii) The pattern suggests the relationship,

$$\sum_{1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1} \quad \text{for} \quad n \geqslant 1$$

Prove by induction the suggested relationship.

[7 marks]

Question 2

Prove by induction the well known result that,

$$\sum_{1}^{n} r = \frac{1}{2} n(n+1)$$

[6 marks]

Question 3

Further A-Level Examination Question from January 2013, FP1, Q8 (a) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

[6 marks]

Question 4

Further A-Level Examination Question from January 2017, IAL, F1, Q9 (i) (Edexcel)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} \left(4r^3 - 3r^2 + r\right) = n^3(n + 1)$$

[6 marks]

Question 5

Further A-Level Examination Question from January 2012, FP1, Q6 (Edexcel)

(a) Prove by induction

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$$

[5 marks]

(b) Using the result in part (a), show that

$$\sum_{r=1}^{n} (r^3 - 2) = \frac{1}{4} n\left(n^3 + 2n^2 + n - 8 \right)$$

[3 marks]

(c) Calculate the exact value of $\displaystyle\sum_{r=20}^{50} \left(r^3 - 2 \right)$

[3 marks]