3.1 The Method of Differences

Thus far, when the Method of Differences has been employed, the differences have occurred somewhat accidentally as a result of applying partial fractions to problems. The concept can be more purposefully set up and deployed to powerful effect.

First, however, a formal statement of exactly what the method of differences is:

The Method of Differences

If the general term, u_r , of a series can be expressed in the form

$$f(r+1) - f(r)$$

then

$$\sum_{r=1}^{n} u_{r} = \sum_{r=1}^{n} (f(r+1) - f(r))$$

In consequence,
$$u_1 = f(2) - f(1)$$

$$u_2 = f(3) - f(2)$$

$$u_3 = f(4) - f(3)$$

$$u_n = f(n+1) - f(n)$$

$$\therefore \sum_{r=1}^{n} u_r = f(n+1) - f(1)$$



3.2 Example

Use the method of differences to find a formula for,

$$\sum_{r=1}^{n} r(r+1)(r+2)$$

Begin by showing that,

$$r(r+1)(r+2) = \frac{1}{4} (r(r+1)(r+2)(r+3) - (r-1)r(r+1)(r+2))$$

Teaching Video: http://www.NumberWonder.co.uk/v9097/3.mp4



3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 50

Question 1

(a) Prove that r(r + 1) - (r - 1)r = 2r

[2 mark]

(**b**) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} r = \frac{1}{2} n (n + 1)$$

Use the facts that

•
$$(r+1)^3 - r^3 = 3r^2 + 3r + 1$$

• $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ (Proven in Question 1)
• $\sum_{r=1}^{n} 1 = n$ (Obvious!)

to construct a proof, using the method of differences, that

$$\sum_{r=1}^{n} r^{2} = \frac{1}{6} n (n+1) (2n+1)$$

Further A-Level Examination Question from June 2016, FP2, Q2

(a) Show that,

$$r^{2}(r+1)^{2} - (r-1)^{2}r^{2} \equiv 4r^{3}$$

[3 marks]

Given that $\sum_{r=1}^{n} r = \frac{1}{2} n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$(1^3 + 2^3 + 3^3 + \dots + n^3) = (1 + 2 + 3 + \dots + n)^2$$

This question is about finding a formula for the series,

$$\sum_{r=1}^{n} (r \times r!) = 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n!$$

(a) Prove that $((r+1)! - 1) - (r! - 1) = r \times r!$

[2 marks]

($\bf b$) Use the part (a) answer and the method of differences to find a formula

for
$$\sum_{r=1}^{n} (r \times r!)$$

[4 marks]

(c) Evaluate
$$\sum_{r=1}^{6} (r \times r!)$$

[1 mark]

This question is about finding a formula for the series,

$$\sum_{r=1}^{n} (r^2 + 1)r! = 2 \times 1! + 5 \times 2! + 10 \times 3! + \dots + (n^2 + 1)n!$$

(a) Prove that
$$((r+2)! - (r+1)!) - 2((r+1)! - r!) = (r^2 + 1)r!$$

[4 marks]

(**b**) Use the part (a) answer and the method of differences to find a formula

for
$$\sum_{r=1}^{n} (r^2 + 1)r!$$

Further A-Level Examination Question from June 2011, FP2, Q4 Given that,

$$(2r + 1)^3 = Ar^3 + Br^2 + Cr + 1$$

(a) find the values of the constants A, B and C

[2 marks]

(**b**) Show that,

$$(2r + 1)^3 - (2r - 1)^3 = 24r^2 + 2$$

[2 marks]

(c) Using the result in part (b) and the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

[5 marks]