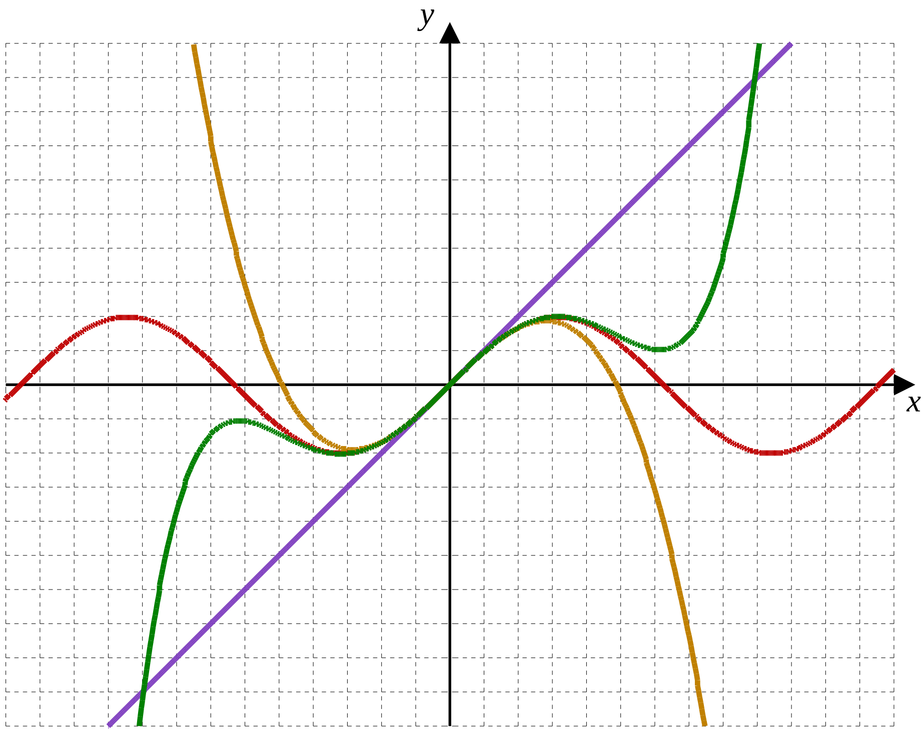
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

MACLAURIN SERIES



~ In search of a polynomial that will model the sine function ~

Purple : $y = x$

Gold : $y = x - \dfrac{x^3}{3!}$

Green : $y = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$

MACLAURIN SERIES

Lesson 1

Further A-Level Pure Mathematics, Core 2
Maclaurin Series

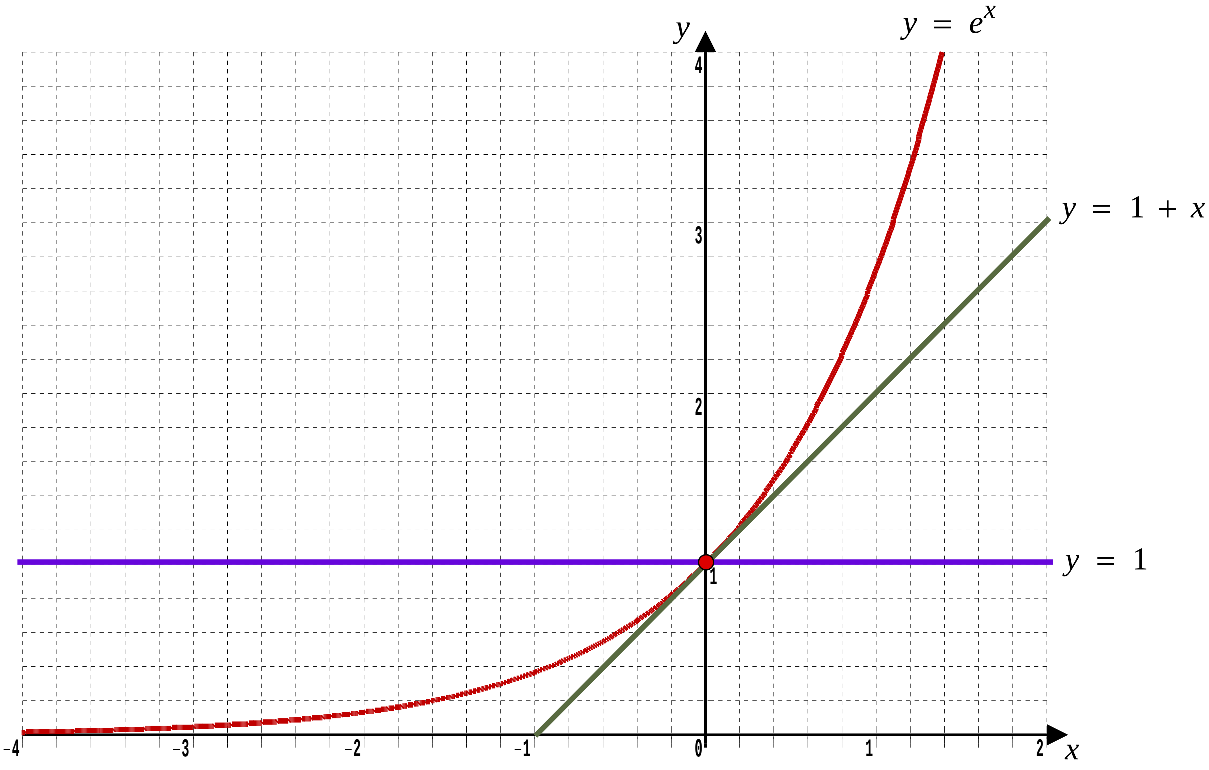
1.1 Approximation

Polynomials first caught our attention because they are straightforward to differentiate or integrate. They have many other properties that make them attractive to work with such as the fact, known as the fundamental theorem of algebra, that a polynomial of degree n has n roots over the complex numbers.

These desirables provide an incentive to approximate other functions, less easy to analyse, with polynomials. As an example, consider the exponential function. What are good polynomial approximations to this curve if we focus on where it crosses the y-axis ?

Here is how we could work up to an answer;

- The best polynomial of degree 0 is $y = 1$
- The best polynomial of degree 1 is $y = 1 + x$



The first approximation, $y = 1$, got the height where $y = e^x$ crossed the y-axis correct. The second approximation, $y = 1 + x$, got both the height the gradient of the exponential curve correct at that y-axis crossing point.
The third approximation will need to get point, gradient and bend correct at that key point (0, 1). Intuitively, a best quadratic curve is sought that will model the exponential and it should be the most successful yet at following the exponential curve for those points that are not too far away from (0, 1).

1.2 The Third Approximation

The exponential function has the remarkable property of being its own derivative.

$$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x \quad \Rightarrow \quad \frac{d^2y}{dx^2} = e^x$$

and so, when $x = 0$, $y = 1$, $\dfrac{dy}{dx} = 1$ and $\dfrac{d^2y}{dx^2} = 1$

Let the best quadratic approximation be,

$$y = a_0 + a_1 x + a_2 x^2$$

It is required that $y = 1$ when $x = 0$ and so $a_0 = 1$

Differentiating the best quadratic approximation gives,

$$\frac{dy}{dx} = a_1 + 2 a_2 x$$

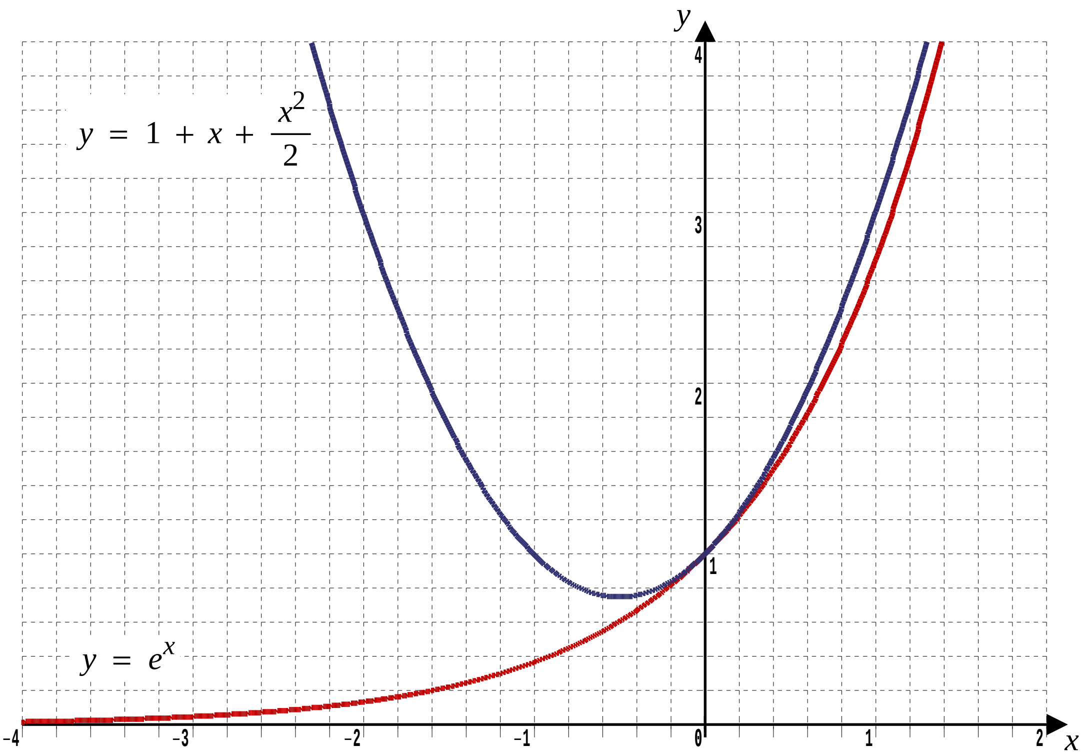
It is required that $\dfrac{dy}{dx} = 1$ when $x = 0$ and so $a_1 = 1$

Differentiating the best quadratic approximation twice gives.

$$\frac{d^2y}{dx^2} = 2 a_2$$

It is required that $\dfrac{d^2y}{dx^2} = 1$ when $x = 0$ and so $a_2 = \dfrac{1}{2}$

The third approximation is thus, $y = 1 + x + \dfrac{x^2}{2}$



1.3 The Fourth Approximation

The approximations studied so far are,

$$y_0 = 1$$

$$y_1 = 1 + x$$

$$y_2 = 1 + x + \frac{x^2}{2}$$

Notice that each higher order approximation contains those previous to it.
In consequence, for the fourth approximation the working can be shortened;

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

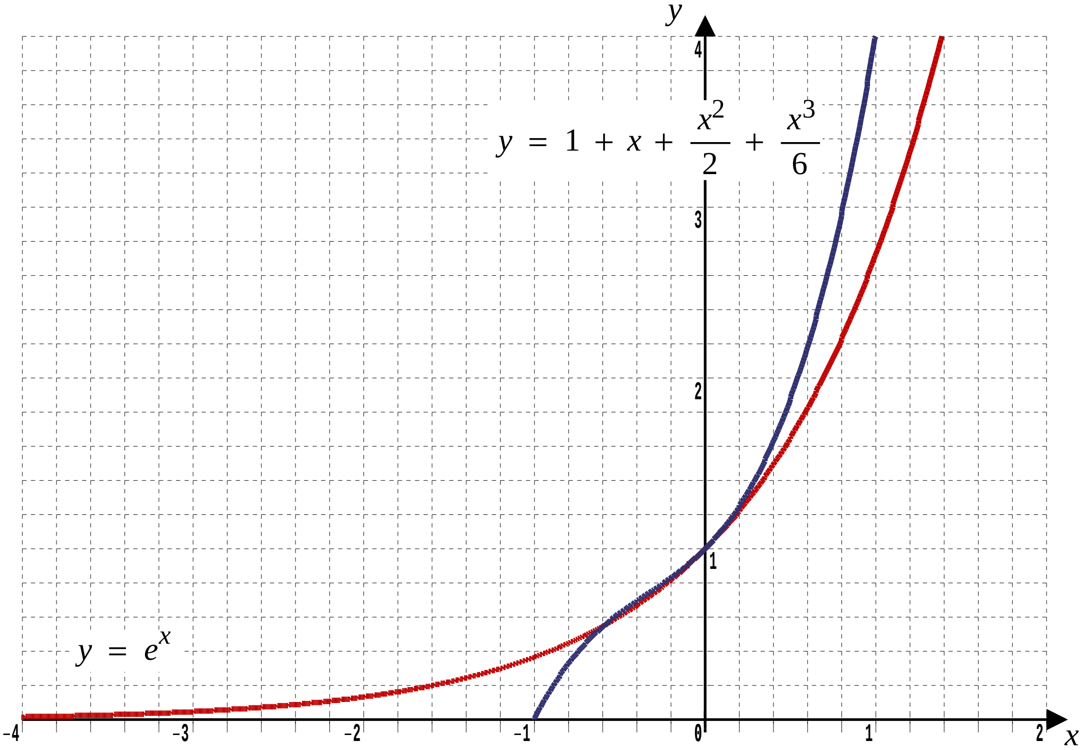
$$\frac{dy}{dx} = \ldots + 3\, a_3 x^2$$

$$\frac{d^2y}{dx^2} = \ldots + 6\, a_3 x$$

$$\frac{d^3y}{dx^3} = \ldots + 6\, a_3$$

It is required that $\dfrac{d^3y}{dx^3} = 1$ when $x = 0$ and so $a_3 = \dfrac{1}{6}$

The fourth approximation is thus, $y = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$



Continuing likewise, the exponential function about (0, 1) can be approximated ever more accurately by taking more terms from the infinite series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad x \in \mathbb{R}$$

1.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

A general polynomial of degree four, a quartic, can be expressed in the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

Complete the table to show the first, second, third and fourth derivatives of y

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$
$\dfrac{dy}{dx} =$
$\dfrac{d^2y}{dx^2} =$
$\dfrac{d^3y}{dx^3} =$
$\dfrac{d^4y}{dx^4} =$

[4 marks]

Question 2

Complete the table to show the first, second, third and fourth derivatives of the function $y = \cos x$ and also the values of those derivatives at $x = 0$

$y = \cos x$	when $x = 0$,	$y = 1$
$\dfrac{dy}{dx} =$	when $x = 0$,	$\dfrac{dy}{dx} =$
$\dfrac{d^2y}{dx^2} =$	when $x = 0$,	$\dfrac{d^2y}{dx} =$
$\dfrac{d^3y}{dx^3} =$	when $x = 0$,	$\dfrac{d^3y}{dx^3} =$
$\dfrac{d^4y}{dx^4} =$	when $x = 0$,	$\dfrac{d^4y}{dx^4} =$

[4 marks]

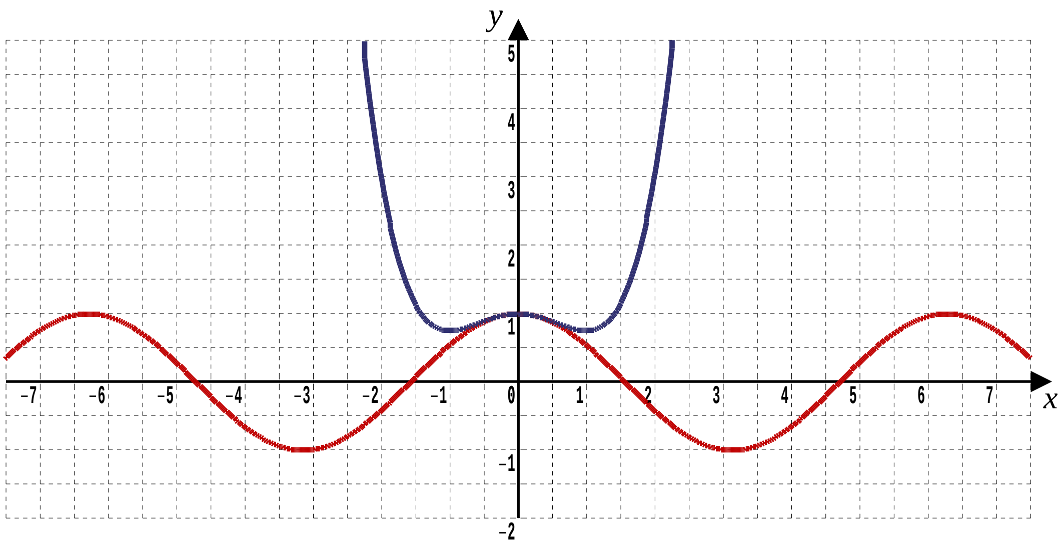
Question 3

By combining the tables of question 1 and 2, work out, in the following order, a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quartic polynomial to represent the cosine function.

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the cosine curve as another (red).

Remember that, as this mathematics is a mixture of calculus and trigonometry, the units of angle must be radians.

Here is an image of what my graph plotter gives;



[6 marks]

Question 4

A general polynomial of degree five, a quintic, can be expressed in the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Complete the table to show the first, second, third, fourth and fifth derivatives of y

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$
$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$
$\frac{d^5y}{dx^5} =$

[4 marks]

Question 5

Complete the table to show the first, second, third, fourth and fifth derivatives of the function $y = sin\, x$ and also the values of those derivatives at $x = 0$

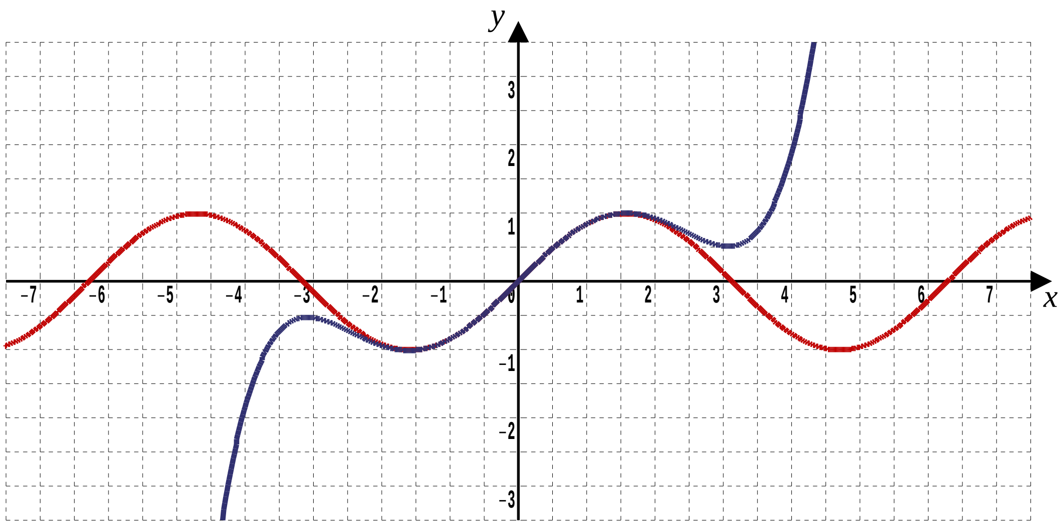
$y = sin\, x$	when $x = 0$,	$y = 0$
$\frac{dy}{dx} =$	when $x = 0$,	$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$	when $x = 0$,	$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$	when $x = 0$,	$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$	when $x = 0$,	$\frac{d^4y}{dx^4} =$
$\frac{d^5y}{dx^5} =$	when $x = 0$,	$\frac{d^5y}{dx^5} =$

[5 marks]

Question 6

By combining the tables of questions 4 and 5, work out, in the following order, a_5 then a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quintic polynomial to represent the sine function.

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the sine curve as another (red).



[6 marks]

Question 7

Taking care over the minus signs, and making use of the chain rule, complete the table to show the first, second, third, fourth and fifth derivatives of the function $y = ln(1 - x)$ where $x < 1$ and also the values of those derivatives at $x = 0$

$y = ln(1 - x)$	when $x = 0$,	$y = 0$
$\frac{dy}{dx} =$	when $x = 0$,	$\frac{dy}{dx} =$
$\frac{d^2y}{dx^2} =$	when $x = 0$,	$\frac{d^2y}{dx^2} =$
$\frac{d^3y}{dx^3} =$	when $x = 0$,	$\frac{d^3y}{dx^3} =$
$\frac{d^4y}{dx^4} =$	when $x = 0$,	$\frac{d^4y}{dx^4} =$
$\frac{d^5y}{dx^5} =$	when $x = 0$,	$\frac{d^5y}{dx^5} =$

[5 marks]

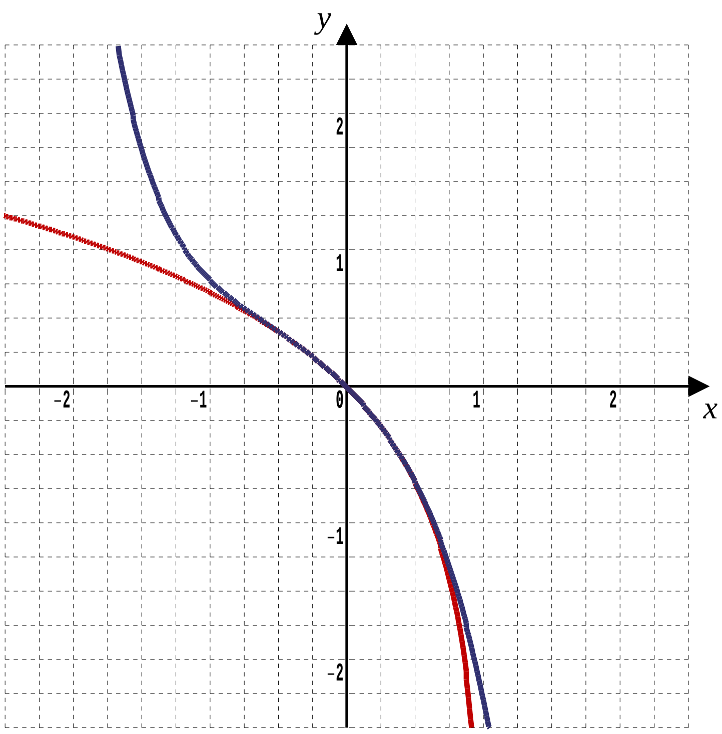
Question 8

$$y = ln(1 - x) \text{ is to be approximated by}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

By combining the tables of questions 4 and 7 work out, in the following order, a_5 then a_4 then a_3 then a_2 then a_1 and lastly a_0. Finally, write out the equation of the best quintic polynomial to represent the function $y = ln(1 - x)$

If you have a graphics calculator, you may like to plot the answer to this question as one curve (blue) and the curve $y = (1 - x)$ as another (red)



[6 marks]