3.1 Making Use of Exponential Form

Although it cannot be assumed that the laws of indices work in the same way with complex numbers as they do with real numbers, one of the merits of writing complex numbers in exponential form is the following key result,

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The Multiply Divide Rule

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

3.2 Example

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then, by means of Euler's Relation, and without using the laws of indices, prove that $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Proof

$$\text{LHS} = \frac{z_1}{z_2}$$

$$= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \qquad \text{using Euler's Relation}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i\cos\theta_2\sin\theta_2 + i\sin\theta_2\cos\theta_2 - i^2\sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}$$

$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \qquad \text{using Euler's Relation}$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Teaching Video : http://www.NumberWonder.co.uk/v9099/3.mp4

The video will talk through the proof on the previous page

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then, by means of Euler's Relation. and without using the laws of indices, prove that $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

[5 marks]

Question 2

If $z = e^{\mathrm{i}\theta}$ then, by means of Euler's Relation, and without using any laws of indices, prove that $\dfrac{1}{z} = e^{-\mathrm{i}\theta}$

[5 marks]

Question 3

Use The Multiply Divide Rule to help calculate the following.

Give your simplified answers in the exponential form $z = r\,e^{\mathrm{i}\theta}$ with $-\pi < \theta \leqslant \pi$

(i) $2\sqrt{3}\,e^{\frac{\pi}{2}\mathrm{i}} \times 3\sqrt{3}\,e^{\frac{\pi}{3}\mathrm{i}}$

[4 marks]

(ii) $\dfrac{\sqrt{8}\,e^{\frac{3\pi}{4}\mathrm{i}}}{\sqrt{2}\,e^{-\frac{2\pi}{3}\mathrm{i}}}$

[4 marks]

(iii) $\sqrt{6}\left(cos\left(-\dfrac{\pi}{12}\right) + \mathrm{i}\,sin\left(-\dfrac{\pi}{12}\right)\right) \times \sqrt{3}\left(cos\left(\dfrac{\pi}{3}\right) + \mathrm{i}\,sin\left(\dfrac{\pi}{3}\right)\right)$

[4 marks]

Question 4

Use the exponential form for a complex number to simplify,

$$\frac{(\cos 11\theta + \mathrm{i}\sin 11\theta)(\cos 5\theta + \mathrm{i}\sin 5\theta)}{\cos 7\theta + \mathrm{i}\sin 7\theta}$$

[3 marks]

Question 5

z and w are such that $z = -9 + 3\sqrt{3}\,\mathrm{i}$, $|w| = \sqrt{3}$ and $arg(w) = \dfrac{7\pi}{12}$

Express the following in the form $r\,e^{\mathrm{i}\theta}$ where $-\pi < \theta \leqslant \pi$

(i) z

(ii) w

(iii) zw

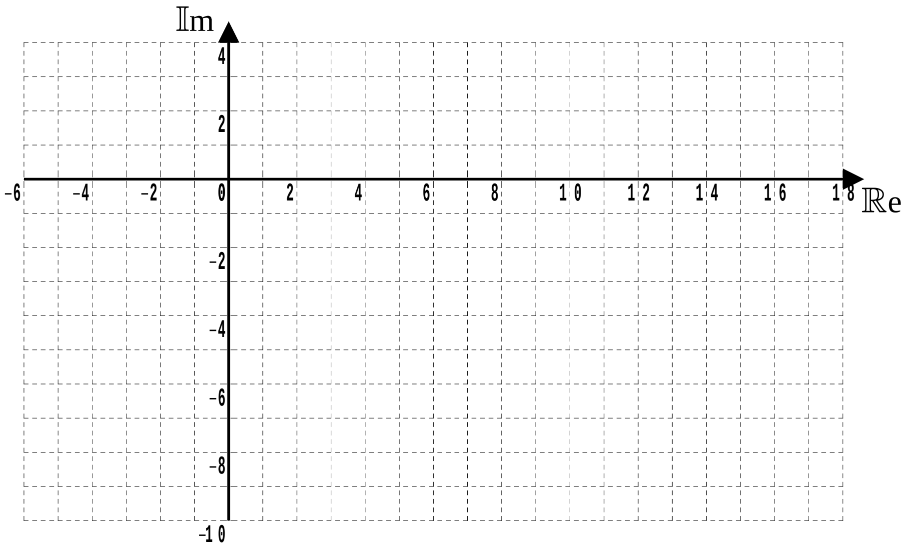
(iv) $\dfrac{z}{w}$

[3, 3, 3, 3 marks]

Question 6

Given that $z = \sqrt{2}\, e^{\frac{\pi}{4}\mathrm{i}}$ write down, in exponential form z^n for n taking all integer values between 1 and 8 inclusive. Convert each power in the form $x + \mathrm{i}\,y$ and plot each of the resulting points on an Argand diagram.
There is no need to cancel down any fractions, or restrict θ to the range $-\pi < \theta \leqslant \pi$

n	z^n (exponential form) $r\,e^{\mathrm{i}\theta}$	z^n (Cartesian form) $x + \mathrm{i}\,y$
1		
2		
3		
4		
5		
6		
7		
8		



What geometric interpretation can be attached to the complex number z ?

[7 marks]

Question 7

This question looks at a proof of Euler's Relation.
It begins by “pulling a rabbit out of a hat” in the form of defining this function;

$$f(\theta) = \frac{\cos\theta + \mathrm{i}\sin\theta}{e^{\mathrm{i}\theta}}$$

(i) Differentiate this function using the quotient rule and show that, for all values of θ, it is zero.

[4 marks]

(ii) If the derivative of a function is zero then the function must equal a constant. To work out what that constant must be, put zero into the original function. State the value of this constant.

[1 mark]

(iii) Explain how Euler's relation may now be deduced.

[1 mark]

Note :
There is an assumption in this proof that the derivative of the exponential function of a complex number behaves as you would hope !