## 6.1 Trigonometric Identities, Type 2

Previously, identities were derived where a trigonometric function was applied to a multiple of the angle $\theta$ and it was equated to an expression containing powers of trigonometric functions in $\theta$.

For example,

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta \qquad \text{(Type 1)}$$

This can be done the other way around, by which is meant a power of a trigonometric function in $\theta$ is equated to a power free expression of trigonometric functions of multiples of the angle $\theta$.

For example,

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) \qquad \text{(Type 2)}$$

## 6.2 A Pair of Essentials

In finding such Type 2 identities the following pair of results is invaluable,

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**The Type 2 Key**

- $z^n + \dfrac{1}{z^n} = 2\cos(n\theta)$
- $z^n - \dfrac{1}{z^n} = 2\,\mathrm{i}\sin(n\theta)$

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*Proof*

$$\begin{aligned}
\text{LHS} &= z^n + \frac{1}{z^n} \\
&= z^n + z^{-n} \\
&= (\cos\theta + \mathrm{i}\sin\theta)^n + (\cos\theta + \mathrm{i}\sin\theta)^{-n} \\
&= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta) && \text{by de Moivre's theorem} \\
&= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) - \mathrm{i}\sin(n\theta) && \text{even and odd functions} \\
&= 2\cos(n\theta) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

A proof of the other result is very similar and is left as an exercise for the reader.

**6.3 Example**

Use the appropriate Type 2 Key result to prove that,

$$cos^4 \theta = \frac{1}{8}(\,cos\,4\theta + 4\,cos\,2\theta + 3\,)$$

Teaching Video : http://www.NumberWonder.co.uk/v9099/6.mp4

**[ 4 marks ]**

**6.4 Exercise**

*Any solution based entirely on graphical*
*or numerical methods is not acceptable*

Marks Available : 50

**Question 1**

The binomial theorem gives that $\left(z + \frac{1}{z}\right)^2 = \left(z^2 + \frac{1}{z^2}\right) + 2$

**( i )** Make use of the "The Type 2 Key" to obtain a useful formula for $cos^2 \theta$

**[ 2 marks ]**

**( ii )** Hence determine the exact value of $\int_0^{\frac{\pi}{2}} cos^2 \theta \, d\theta$

**[ 2 marks ]**

**( iii )** In a similar manner, consider $\left(z - \frac{1}{z}\right)^2$ and use "The Type 2 Key"

to determine the exact value of $\int_0^{\frac{\pi}{6}} sin^2 \theta \, d\theta$

**[ 3 marks ]**

**Question 2**

**( i )** Beginning with an expression for $\left(z - \frac{1}{z}\right)^3$ find the constants $p$ and $q$ in the identity $sin^3\,\theta = p\,sin\,3\theta + q\,sin\theta$

**[ 4 marks ]**

**( ii )** Prove $sin\left(\frac{\pi}{2} - \theta\right) = cos\,\theta$ and $sin\left(\frac{3\pi}{2} - 3\theta\right) = -cos\,3\theta$

**[ 2 marks ]**

**( iii )** Combine your part (i) and (ii) results to find the constants $r$ and $s$ in the identity $cos^3\,\theta = r\,cos\,3\theta + s\,cos\,\theta$

**[ 2 marks ]**

**Question 3**

**( i )** Express $\left(z^2 + \frac{1}{z^2}\right)^3$ in terms of $cos\,6\theta$ and $cos\,2\theta$

**[ 3 marks ]**

**( ii )** Hence find constants *a* and *b* such that $cos^3\,2\theta = a\,cos\,6\theta + b\,cos\,2\theta$

**[ 3 marks ]**

**( iii )** Hence show that $\int_0^{\frac{\pi}{6}} cos^3\,2\theta\,d\theta = k\sqrt{3}$ where *k* is a rational constant

**[ 4 marks ]**

**Question 4**

*Further A-Level Examination Question from June 2016, FP2, Q5 (Edexcel)*

**( a )** Use de Moivre's theorem to show that

$$\sin^5 \theta \equiv a \sin 5\theta + b \sin 3\theta + c \sin \theta$$

where $a$, $b$ and $c$ are constants to be found

**[ 5 marks ]**

**( b )** Hence show that $\int_0^{\frac{\pi}{3}} \sin^5 \theta \, d\theta = \frac{53}{480}$

**[ 5 marks ]**

**Question 5**

*Further A-Level Examination Question from June 2007, FP3, Q4(b)(c) (Edexcel)*

**( i )** Express $32\,cos^6\,\theta$ in the form $p\,cos\,6\theta + q\,cos\,4\theta + r\,cos\,2\theta + s$ where $p$, $q$, $r$ and $s$ are integers

**[ 5 marks ]**

**( ii )** Hence find the exact value of $\int_0^{\frac{\pi}{3}} cos^6\,\theta\,d\theta$

**[ 4 marks ]**

**Question 6**

*Further A-Level Examination Question from June 2006, FP2, Q2(a)(ii) (MEI)*

By considering $\left(z - \frac{1}{z}\right)^4\left(z + \frac{1}{z}\right)^2$ find $A$, $B$, $C$ and $D$ such that

$$sin^4\,\theta\,cos^2\,\theta \;=\; A\cos 6\theta \;+\; B\cos 4\theta \;+\; C\cos 2\theta \;+\; D$$

**[ 6 marks ]**