7.1 Complex Number Polygons

The focus of this lesson is upon solving equations of the form $z^n = w$ where z and w are complex numbers and n is a positive integer.

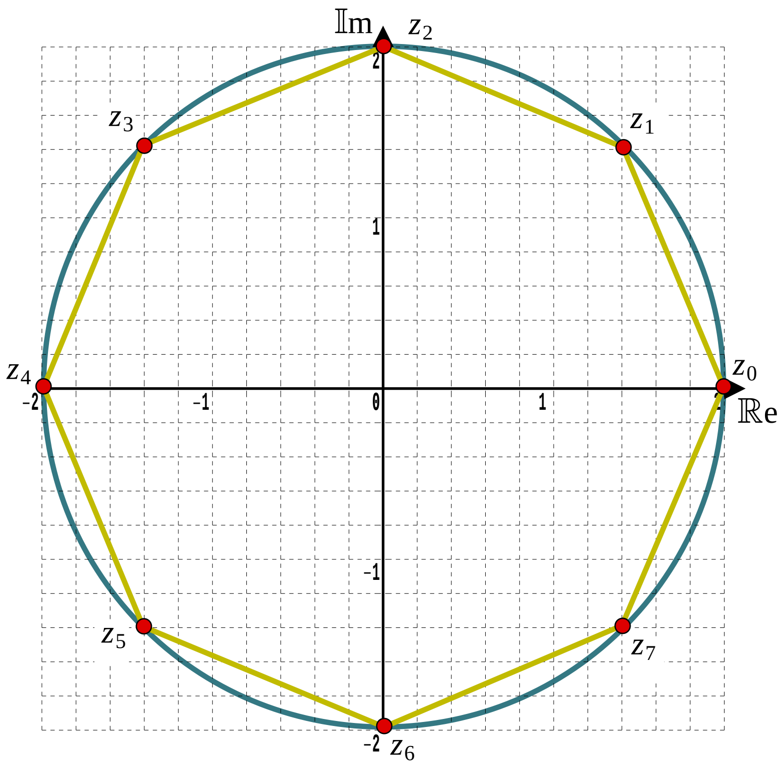
The Fundamental Theorem of Algebra states that over the complex numbers a polynomial of degree n will have n roots. For any given n and w the task is to then find all n roots.

When the equation $z^8 = 256$ is solved over the real numbers, two answers are are obtained, $z = \pm 2$, because they are the only two real numbers that make the equation true.

Over the complex numbers, however, the same equation has these eight roots;

$$z \in \{\pm 2, \pm 2\text{i}, \pm(\sqrt{2} + \sqrt{2}\,\text{i}), \pm(\sqrt{2} - \sqrt{2}\,\text{i})\}$$

On an Argand diagram this collection of roots lie at the vertices of a regular octagon with its centre at the origin.



Thus there is thus a natural relationship between the roots of equations of the form $z^n = w$ and a regular n-gon. The radius of the circle that circumscribes the n-gon and the angle by which the n-gon is rotated about the origin depend upon w.

7.2 Root Finding

Complex Roots Theorem

For a positive integer *n*, $w = r(\cos\theta + \mathrm{i}\sin\theta)$ has exactly *n* distinct n^{th} roots given by

$$z_k = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + \mathrm{i}\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

where $k = 0, 1, 2, 3, ..., n - 1$

On an Argand diagram the roots lie at the vertices of a regular *n*-gon with its centre at the origin.

Example

To solve the equation $z^8 = 256$ with the answers in Cartesian form...

Begin by observing that $r = 256$, $\theta = 0$ and so $w = 256$ is better expressed in trigonometric form as $w = 256(\cos 0 + \mathrm{i}\sin 0)$

From the Complex Roots Theorem, the roots will be given by,

$$z_k = \sqrt[8]{256}\left(\cos\left(\frac{0 + 2\pi k}{8}\right) + \mathrm{i}\sin\left(\frac{0 + 2\pi k}{8}\right)\right)$$

$$= 2\left(\cos\left(\frac{\pi k}{4}\right) + \mathrm{i}\sin\left(\frac{\pi k}{4}\right)\right) \text{ for } k = 0, 1, 2, ..., n - 1$$

For $k = 0 : z_0 = 2(\cos 0 + \mathrm{i}\sin 0) = 2$

For $k = 1 : z_1 = 2\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2} + \sqrt{2}\,\mathrm{i}$

For $k = 2 : z_2 = 2\left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right) = 2\,\mathrm{i}$

For $k = 3 : z_3 = 2\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right) = -\sqrt{2} + \sqrt{2}\,\mathrm{i}$

For $k = 4 : z_4 = 2(\cos(\pi) + \mathrm{i}\sin(\pi)) = -2$

For $k = 5 : z_5 = 2\left(\cos\left(\frac{5\pi}{4}\right) + \mathrm{i}\sin\left(\frac{5\pi}{4}\right)\right) = -\sqrt{2} - \sqrt{2}\,\mathrm{i}$

For $k = 6 : z_6 = 2\left(\cos\left(\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{2}\right)\right) = -2\,\mathrm{i}$

For $k = 7 : z_7 = 2\left(\cos\left(\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(\frac{7\pi}{4}\right)\right) = \sqrt{2} - \sqrt{2}\,\mathrm{i}$

These are the roots shown on the diagram on the previous page

7.3 A Special Case

Considerable importance is attached to the special case of solving the equation $z^n = w$ when $w = 1$. The resulting roots are termed the n^{th} roots of unity. Geometrically, on an Argand diagram the roots will be equally spaced around a unit circle, with one root always at $1 + 0\text{i}$

The n^{th} Roots of Unity

The n^{th} roots of unity are the roots of the equation $z^n = 1$

If those roots are denoted as $1, \omega, \omega^2, \ldots, \omega^{n-1}$ then those roots sum to zero

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$$

Proof #1

The n^{th} roots of unity are the solutions to the equation $z^n - 1 = 0$.

In this equation the coefficient of z^n is 1 and the coefficient of z^{n-1} is zero.

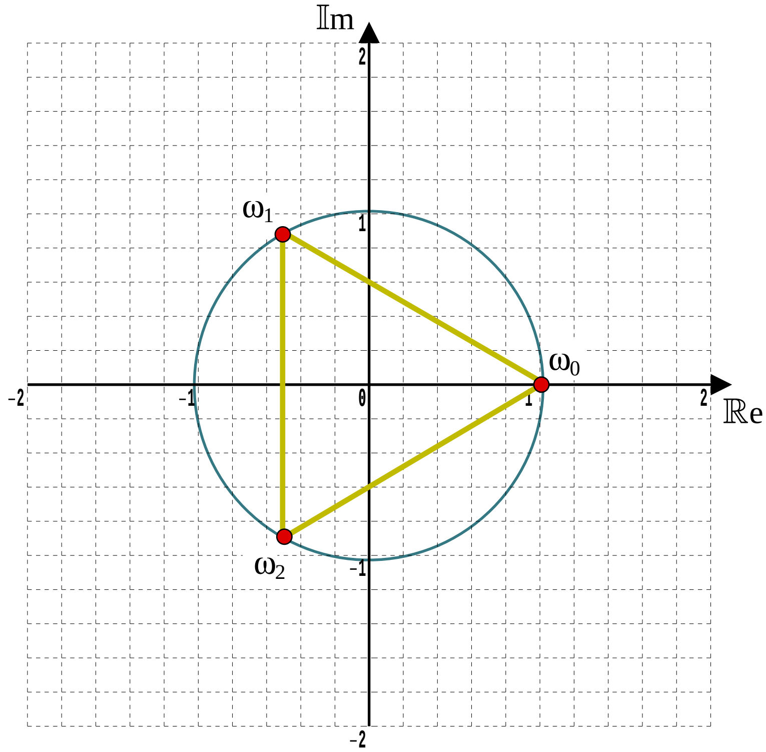
The sum of the roots of a polynomial is given by the coefficient of z^{n-1} □

Proof #2

Geometrically, the n^{th} roots of unity are equally spaced vectors around a unit circle and so their sum is the centre of the circle which is $0 + 0\text{i}$ □

Example

The cube roots of unity are $1, \quad -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\text{i}$



7.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

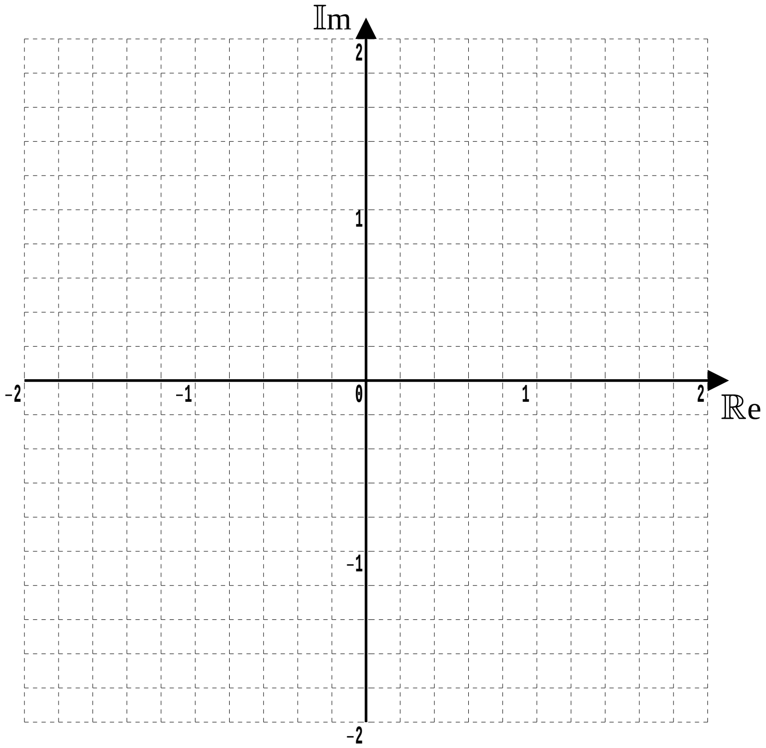
Marks Available : 50

Question 1

(i) Solve the equation $z^3 = \frac{27}{8}\text{i}$ presenting the exact answers in Cartesian form.

[6 marks]

(ii) Plot the part (i) answers on an Argand diagram. By considering them the vertices of an equilateral triangle, determine the exact area of the triangle.



[5 marks]

Question 2

Further A-Level Examination Question from June 2009, FP2, Q2 (Edexcel)

Solve the equation

$$z^3 = 4\sqrt{2} - 4\sqrt{2}\,\mathrm{i}$$

giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$

[6 marks]

Question 3
Further A-Level Examination Question from June 2017, FP2, Q3 (Edexcel)
Solve the equation,

$$z^3 + 32 + 32\mathrm{i}\sqrt{3} = 0$$

giving your answers in the form $r\,e^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$

[6 marks]

Question 4

(i) Find the five roots of the equation $z^5 - 1 = 0$

Give your answers in the form $r\,(\,cos\,\theta + i\,sin\,\theta\,)$ where $-\pi < \theta \leqslant \pi$

[6 marks]

(ii) Hence, or otherwise, show that,

$$cos\left(\frac{2\pi}{5}\right) + cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$$

[3 marks]

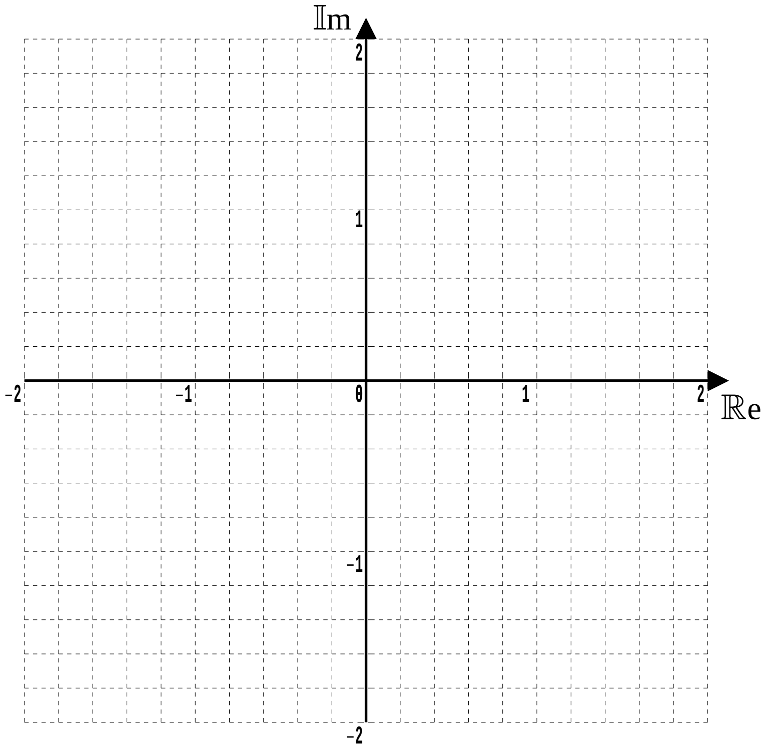
Question 5

(i) Find the three roots of the equation $(z + 1)^3 = -1$

Give your answers in the form $x + iy$, where $x, y \in \mathbb{R}$

[6 marks]

(ii) Plot the points representing these three roots on an Argand diagram.



[2 marks]

(iii) Given that these three points lie on a circle, state its centre and radius

[1 mark]

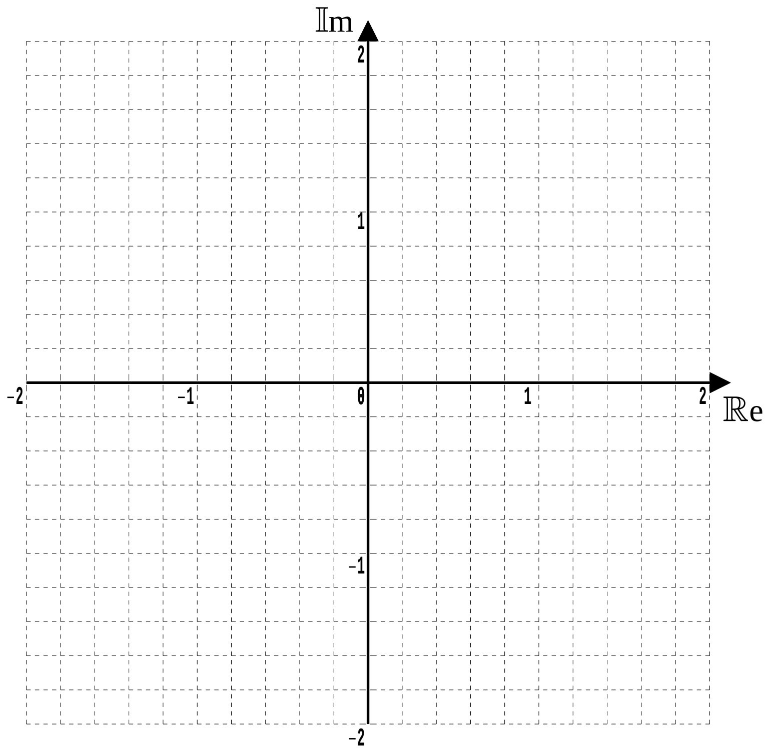
Question 6

Further A-Level Examination Question from June 2016, FP2, Q3 (Edexcel)

(a) Find the four roots of the equation $z^4 = 8(\sqrt{3} + \mathrm{i})$ in the form $z = re^{\mathrm{i}\theta}$

[5 marks]

(b) Show these roots on an Argand diagram



[2 marks]