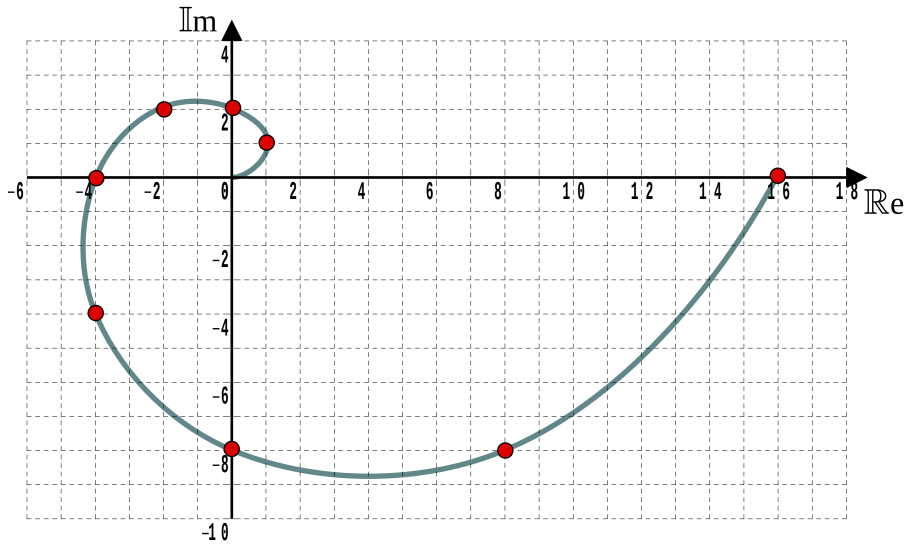
8.1 The Spiral Similarity

In lesson 3† the effect of repeatedly multiplying the point $(1 + \mathrm{i})$ by the complex number $(1 + \mathrm{i})$ was investigated. Although the calculation can be performed with the point and the multiplying number both in Cartesian form, the use of exponential form made both the calculation easier, and clarified the underlying structure of the process, especially by not simplifying fractions or insisting that θ be in the range $-\pi < \theta \leqslant \pi$

n	z^n (exponential form) $r\,e^{\mathrm{i}\theta}$	z^n (Cartesian form) $x + \mathrm{i}\,y$
1	$\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}$	$1 + \mathrm{i}$
2	$2\,e^{\frac{2\pi}{4}\mathrm{i}}$	$0 + 2\,\mathrm{i}$
3	$2\sqrt{2}\,e^{\frac{3\pi}{4}\mathrm{i}}$	$-2 + 2\,\mathrm{i}$
4	$4\,e^{\frac{4\pi}{4}\mathrm{i}}$	$-4 + 0\,\mathrm{i}$
5	$4\sqrt{2}\,e^{\frac{5\pi}{4}\mathrm{i}}$	$-4 - 4\,\mathrm{i}$
6	$8\,e^{\frac{6\pi}{4}\mathrm{i}}$	$0 - 8\,\mathrm{i}$
7	$8\sqrt{2}\,e^{\frac{7\pi}{4}\mathrm{i}}$	$8 - 8\,\mathrm{i}$
8	$16\,e^{\frac{8\pi}{4}\mathrm{i}}$	$16 + 0\,\mathrm{i}$

Plotting the point after each multiplication revealed an attractive spiral lurking in the background;



† Lesson 3, Exercise 3.3, Question 6

Each successive multiplication by $\sqrt{2}\,e^{\frac{\pi}{4}\mathrm{i}}$ rotated the point by $\dfrac{\pi}{4}$ radians and increasing its distance from the origin by a length scale factor of $\sqrt{2}$

This type of transformation, a rotation in combination with an enlargement is termed a spiral similarity.

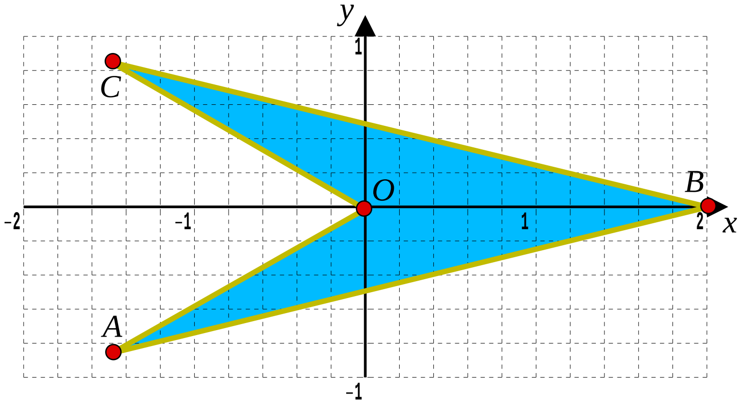
The Spiral Similarity

Multiplication of a point, or a set of points, on an Argand diagram by $r\,e^{\theta\mathrm{i}}$ gives rise to a spiral similarity where all points are rotated anticlockwise about the origin by an angle θ, and their distances from the origin alter by the factor r.

When $r = 1$ the transformation reduces to a rotation by θ about the origin.

8.2 Example

A dart has vertices O, A, B and C. B is the point (2, 0) and A and C are spiral similarities of B rotated by $\pm\dfrac{5\pi}{6}$ radians about the origin and scale factor $\dfrac{\sqrt{3}}{2}$.



Find the exact Cartesian coordinates of A and C and the exact area of the dart.

Teaching Video : http://www.NumberWonder.co.uk/v9099/8.mp4

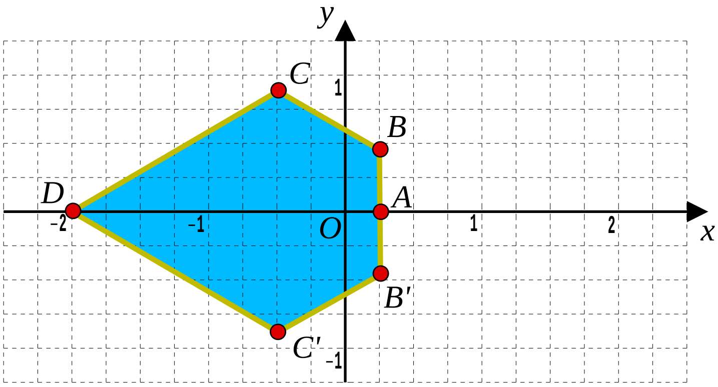
[6 marks]

8.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

An irregular pentagon has vertices B, C, D, C' and B' where C' and B' are reflections in the x-axis of C and B respectively, as shown in the diagram.



A is the midpoint of the vertical line BB' and has coordinates $\left(\frac{1}{4},\ 0\right)$.

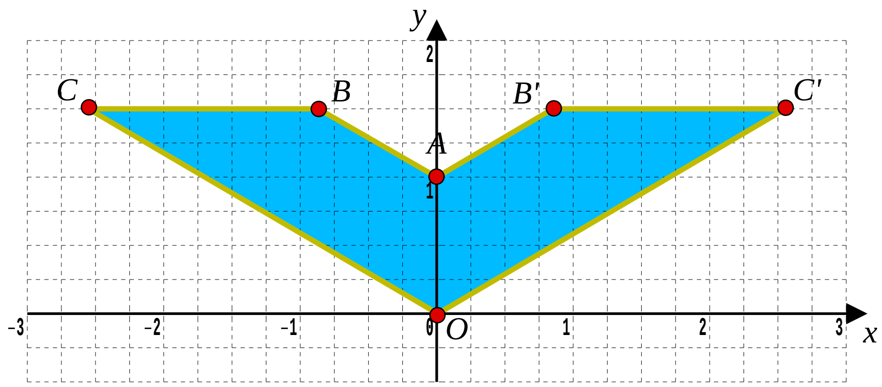
B is a spiral similarity of A rotated by $\frac{\pi}{3}$ radians about O and scale factor 2.

The same spiral similarity applied to B gives C, and applied to C gives D.
Find the exact Cartesian coordinates of B, C and D and the exact area of the pentagon.

[7 marks]

Question 2

An irregular hexagon has vertices *A*, *B*, *C*, *O*, *C′* and *B′* where *C′* and *B′* are reflections in the *y*-axis of *C* and *B* respectively, as shown in the diagram.



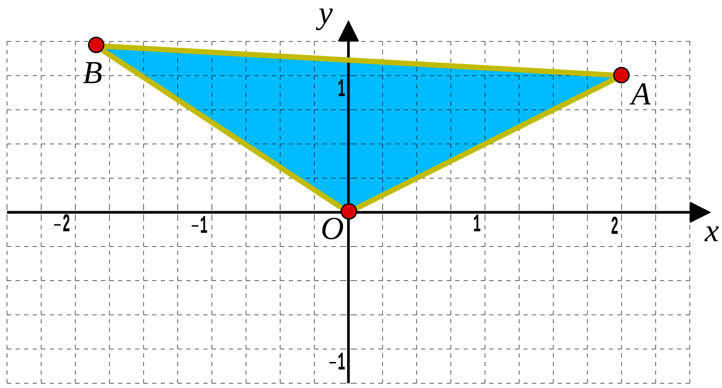
A has coordinates $(0,\ 1)$ and *B* is a spiral similarity of *A* rotated by $\dfrac{\pi}{6}$ radians about *O* with scale factor $\sqrt{3}$. The same spiral similarity applied to *B* gives *C*.
Find the exact coordinates of *B* and *C* and the exact area of the hexagon.

[7 marks]

Question 3

Triangle *OAB* has vertex *O* at the origin and vertex *A* at coordinates (2, 1).

Vertex *B* is a rotation of vertex *A* by $\dfrac{2\pi}{3}$ radians about the origin.



(i) Given that the exact coordinates of vertex *B* are sought, identify the issue that will arise if vertex *A* is written in exponential form.

[2 marks]

(ii) By writing the rotation in the Cartesian form $x + \mathrm{i}y$ find the exact coordinates of vertex *B*.

[5 marks]

(iii) Find the exact perimeter of triangle *OAB*

[2 marks]

Question 4

Further A-Level Examination Question from June 2019, Core 2, Q6 (Edexcel)

In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin.

The point A represents the complex number $6 + 2\mathrm{i}$

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact.

[6 marks]

The points D, E and F are the midpoints of the sides of triangle ABC

(b) Find the exact area of triangle DEF

[3 marks]

Question 5

Further A-Level Examination Question from June 2018, FP2, Q2(b) (MEI)

The vertices of a square with sides of length 1 unit lie on the axes on an Argand diagram. The vertices represent the complex numbers z_1, z_2, z_3 and z_4 and the midpoints of the sides of the square represent complex numbers z_5, z_6, z_7 and z_8

(i) Express z_5, z_6, z_7 and z_8 in modulus-argument form, and hence determine a polynomial equation of degree 4, with integer coefficients, whose roots are z_5, z_6, z_7 and z_8

[4 marks]

Let $P(z)$ be a polynomial equation of degree 8, with integer coefficients, whose roots are $z_1, z_2, z_3, z_4, z_5, z_6, z_7$ and z_8

(ii) Explain why $P(z)$ cannot be of the form $a\,z^8 + b$ where a and b are integers

[1 mark]

(iii) Find $P(z)$

[4 marks]

Question 6

Further A-Level Examination Question from January 2013, FP2, Q2(b) (MEI)

(i) Express $e^{\frac{2\pi}{3}\mathrm{i}}$ in the form $x + \mathrm{i}y$ where the real numbers x and y should be given exactly

[1 mark]

An equilateral triangle in the Argand diagram has its centre at the origin. One vertex of the triangle is at the point representing $2 + 4\mathrm{i}$

(ii) Obtain the complex numbers representing the other two vertices, giving your answers in the form $x + \mathrm{i}y$ where the real numbers x and y should be given exactly.

[6 marks]

(iii) Show that the length of a side is $2\sqrt{15}$

[2 marks]