Lesson 9

Further A-Level Pure Mathematics, Core 2

Complex Numbers II

9.1 Complex Geometric Progressions

It has been established over the previous eight lessons that many results from the world of real numbers have corresponding results in the complex numbers. For example, several of the rules of indices carry across, albeit with very different reasoning behind why they still work. One very useful real number result that more directly carries across is that for summing a Geometric Progression.

The Sum of a Geometric Progression

- $\displaystyle\sum_{k=0}^{n-1} a\,r^k = a + ar + a\,r^2 + \ldots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}$
- $\displaystyle\sum_{k=0}^{\infty} a\,r^k = a + ar + a\,r^2 + \ldots = \frac{a}{1 - r},\quad |r| < 1$

where $a,\ r \in \mathbb{C}$

9.2 Two Useful Results

When working with complex numbers that are in Geometric Progression the following pair of results are worth keeping in mind,

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Two Trigonometric Functions, as Exponentials

- $\cos(n\theta) = \dfrac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right)$
- $\sin(n\theta) = \dfrac{1}{2\mathrm{i}}\left(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}\right)$

Proof

$$\begin{aligned}
\text{RHS} &= \frac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right) \\
&= \frac{\cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta)}{2} && \text{using Euler's Relation} \\
&= \frac{\cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) - \mathrm{i}\sin(n\theta)}{2} && \text{even and odd functions} \\
&= \frac{2\cos(n\theta)}{2} \\
&= \cos(n\theta) \\
&= \text{LHS} \qquad \square
\end{aligned}$$

A proof of the other result is very similar and is left as an exercise for the reader.

9.3 Example

Further A-Level Examination Question from January 2007, Q2(b)(ii) (MEI)

Express $e^{-\frac{\theta}{2}i} + e^{\frac{\theta}{2}i}$ in simplified trigonometric form and hence, or otherwise

show that $1 + e^{\theta i} = 2\,e^{\frac{\theta}{2}i}\,cos\left(\frac{\theta}{2}\right)$

Teaching Video : http://www.NumberWonder.co.uk/v9099/9.mp4

[4 marks]

9.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Express $e^{-\frac{\theta}{2}i} - e^{\frac{\theta}{2}i}$ in simplified trigonometric form and hence, or otherwise

show that $e^{\theta i} - 1 = 2i\,e^{\frac{\theta}{2}i}\,sin\left(\frac{\theta}{2}\right)$

[4 marks]

Question 2

The convergent infinite series C and S are defined as,

$$C = 1 + \frac{1}{3} \cos \theta + \frac{1}{9} \cos 2\theta + \frac{1}{27} \cos 3\theta + ...$$

$$S = \frac{1}{3} \sin \theta + \frac{1}{9} \sin 2\theta + \frac{1}{27} \sin 3\theta + ...$$

(i) Show that $C + \mathrm{i}S = \dfrac{3}{3 - e^{\theta \mathrm{i}}}$

[4 marks]

(ii) Hence show that $C = \dfrac{9 - 3\cos\theta}{10 - 6\cos\theta}$ and find a similar expression for S

[4 marks]

Question 3

Further A-Level Examination Question from June 2019, Paper 2, Q4 (Edexcel)

The infinite series *C* and *S* are defined as,

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + ...$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + ...$$

Given that the series *C* and *S* are both convergent,

(a) Show that $C + \mathrm{i}S = \dfrac{2e^{\mathrm{i}\theta}}{2 - e^{4\mathrm{i}\theta}}$

[4 marks]

(ii) Hence show that $S = \dfrac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$

[4 marks]

Question 4

(i) Use Euler's Relation to express $e^{\mathrm{i}n\theta}$ in trigonometric form

[1 mark]

(ii) Use Euler's Relation to express $e^{-\mathrm{i}n\theta}$ in trigonometric form

[1 mark]

(iii) By treating your part (i) and part (ii) answers as a pair of simultaneous equation, and adding them together, prove that,

$$cos(n\theta) = \frac{1}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right)$$

[1 mark]

(iv) In a similar manner, prove a similar result for $sin(n\theta)$

[1 mark]

Question 5

Show that $\displaystyle\sum_{k=0}^{7}(1 + \mathrm{i})^k = -15\mathrm{i}$

[4 marks]

Question 6

(i) Show that $\left(2 + e^{\mathrm{i}\theta}\right)\left(2 + e^{-\mathrm{i}\theta}\right) = 5 + 4\cos\theta$

[2 marks]

The convergent infinite series C and S are defined by

$$C = 1 - \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta - \frac{1}{8}\cos 3\theta + \ldots$$

$$S = \frac{1}{2}\sin\theta - \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots$$

(ii) By considering $C - \mathrm{i}S$ show that $C = \dfrac{4 + 2\cos\theta}{5 + 4\cos\theta}$ and write down the corresponding expression for S

[4 marks]

Question 7

Further A-Level Examination Question from January 2013, FP2, Q2 (MEI)

(i) Show that, $1 + e^{2\theta \mathrm{i}} = 2\cos\theta\,(\cos\theta + \mathrm{i}\sin\theta)$

[2 marks]

(ii) The series C and S are defined as follows,

$$C = 1 + \binom{n}{1}\cos 2\theta + \binom{n}{2}\cos 4\theta + \dots + \cos 2n\theta$$

$$S = \binom{n}{1}\sin 2\theta + \binom{n}{2}\sin 4\theta + \dots + \sin 2n\theta$$

By considering $C + \mathrm{i}S$ show that $C = 2^n \cos^n\theta \cos n\theta$ and find a corresponding expression for S

[7 marks]

Question 8

Determine, in as simple a form as possible, the value of,

(i) $\displaystyle\sum_{k=0}^{3} \mathrm{i}^{k}$

[1 mark]

(ii) $\displaystyle\sum_{k=1}^{15} \mathrm{i}^{k}$

[2 marks]

(iii) $\displaystyle\sum_{k=0}^{12} z^{k}$ given that $z = e^{\frac{\pi}{2}\mathrm{i}}$

[2 marks]

(iv) $\displaystyle\sum_{k=0}^{121} z^{k}$ given that $z = e^{-\frac{\pi}{2}\mathrm{i}}$

[2 marks]