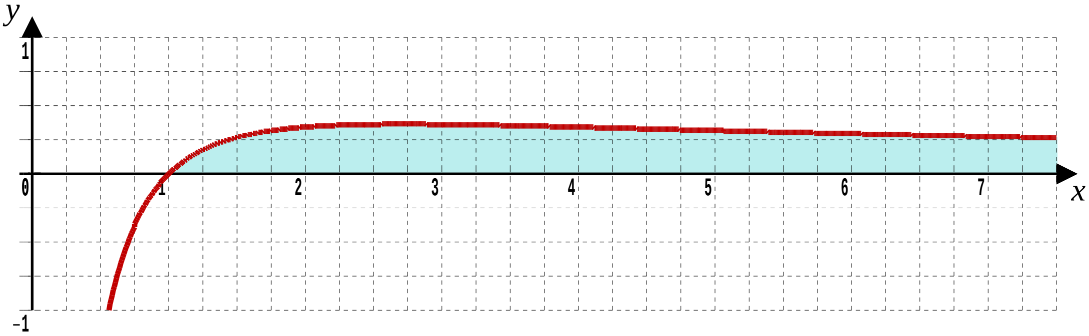
Lesson 2

Further A-Level Pure Mathematics, Core 2
Improper Integrals

2.1 By Parts

The graph is of the function $f(x) = \dfrac{\ln x}{x}$ and the interest is in finding the area shown shaded but extending off to infinity.
From the graph it's not clear if this area will have a finite value.



Teaching Video : http://www.NumberWonder.co.uk/v9100/2.mp4

[6 marks]

Mathematical Folklore
Dr Peter Lax has made outstanding, innovative and profound contributions to the theories of partial differential equation and numerical analysis. When awarded the National Medal of Science in 1986 a newspaper journalist asked what he did to deserve the medal. Lax responded, “I used Integration by Parts”, which was duly reported in the International press at the time.

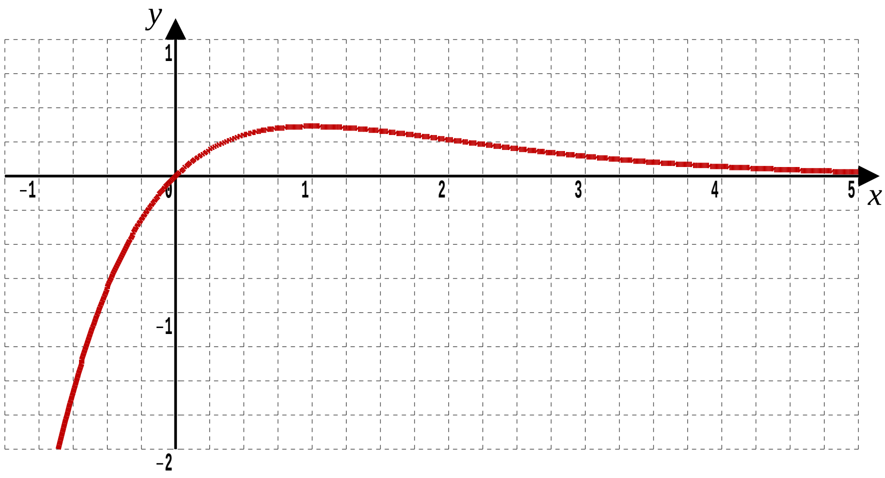
2.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 50

Question 1

The graph is of the function $f(x) = x e^{-x}$



Use integration by parts to determine the value of $\int_0^\infty f(x)\, dx$

You may assume that as $t \to \infty$, $e^{-t} \to 0$ and $t e^{-t} \to 0$

[6 marks]

Question 2

The graph is of the function $f(x) = e^{-x} \sin x$

(i) Explain, briefly, why as $x \to \infty$, $f(x) \to 0$

[1 mark]

(ii) Let $S = \int e^{-x} \sin x \, dx$

Use integration by parts twice to show that

$S = -\frac{1}{2} e^{-x} (\cos x + \sin x) + c$ where c is a constant

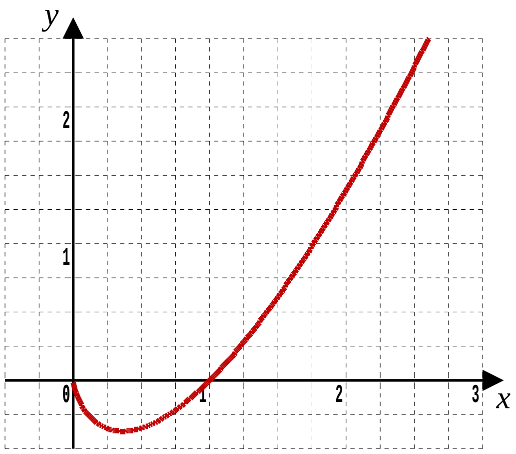
[6 marks]

(iii) Determine the value of $\int_0^{\infty} e^{-x} \sin x \, dx$

[3 marks]

Question 3

The graph is of the function $f(x) = x \ln x$

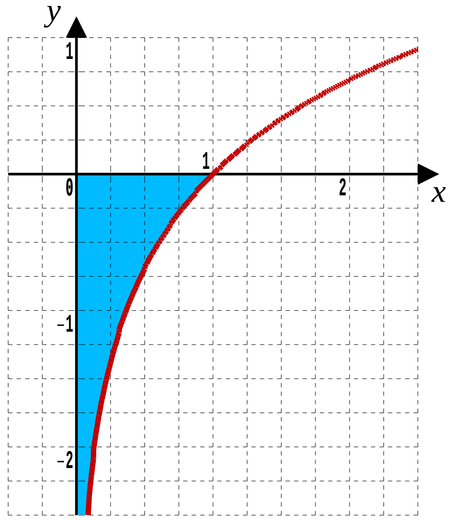


(i) As $x \to 0^{+}$ what does the graph suggest that $x \ln x$ will tend to ?

[1 mark]

Proving the part (i) suggestion requires L'Hôspital's Rule, covered in the “Further Pure 1” optional unit of the Further A-Level Mathematics course, on page 152.

The graph is of the function $g(x) = \ln x$

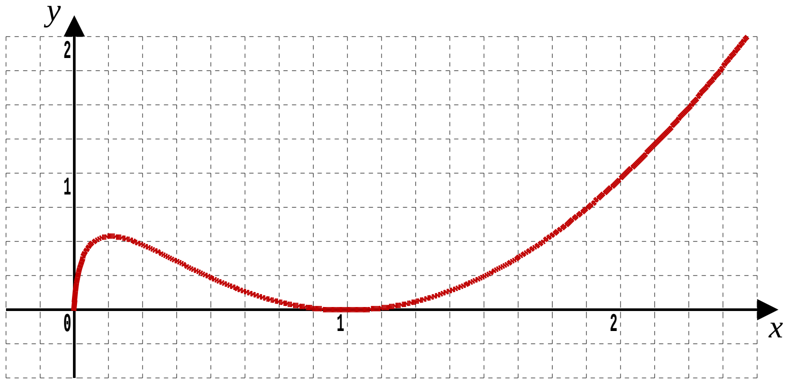


(ii) Find the shaded area enclosed by the graph of the function $g(x) = \ln x$ and the coordinate axes.

[6 marks]

Question 4

The graph is of the function $f(x) = x\,(\ln x)^2$

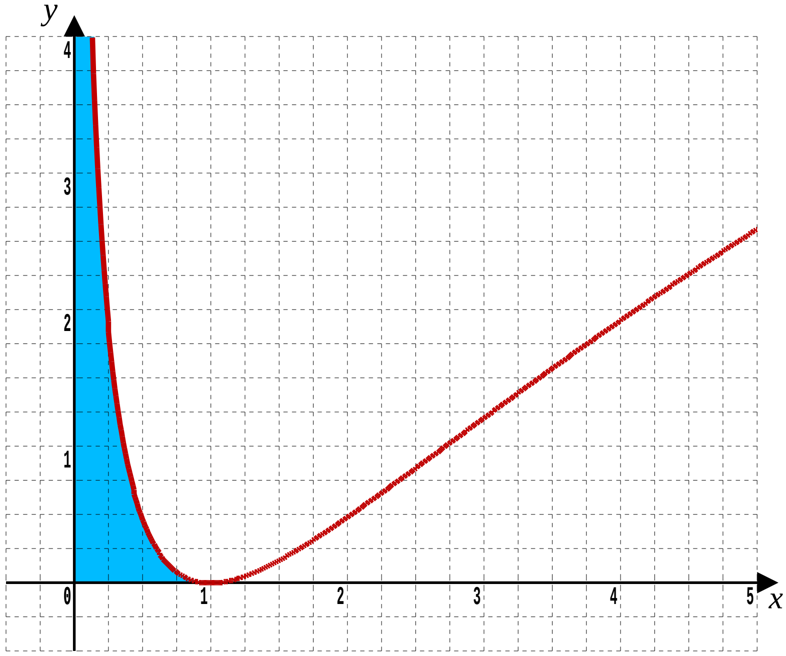


(i) As $x \to 0^+$ what does the graph suggest that $x\,(\ln x)^2$ will tend to ?

[1 mark]

Again, proving the part (i) suggestion requires L'Hôspital's Rule, covered in the "Further Pure 1" optional unit of the Further A-Level Mathematics course.

The graph is of the function $g(x) = (\ln x)^2$



(ii) Find the shaded area enclosed by the graph of the function $g(x) = (\ln x)^2$ and the coordinate axes.

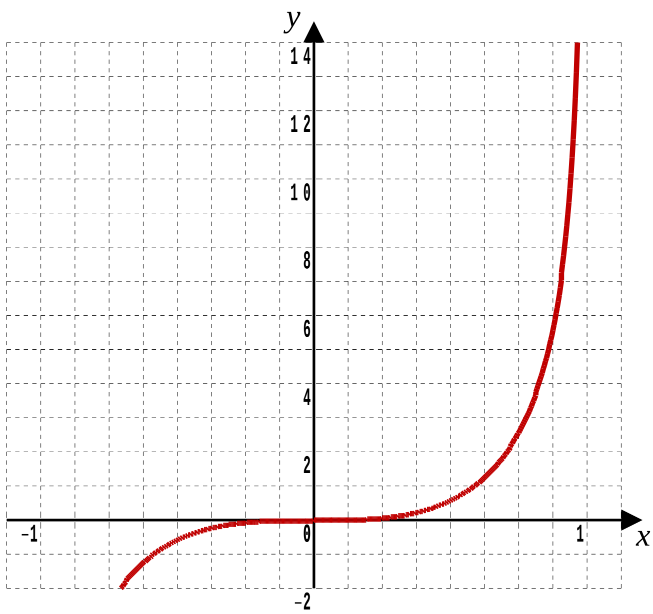
[6 marks]

Question 5

This question is about using integration by parts to show that the following improper integral is convergent, and finding its value,

$$\int_0^1 \frac{x^3}{\sqrt{1 - x^2}}\, dx$$

(i) On the graph, shade in the area to be found,



[1 mark]

(ii) Differentiate each of the following,

(a) $y = \left(1 - x^2\right)^{\frac{3}{2}}$ **(b)** $y = \left(1 - x^2\right)^{\frac{1}{2}}$

[2 marks]

(iii) By rewriting the integral in the following way,

$$\int_0^1 1\frac{x^3}{\sqrt{1 - x^2}}\,dx = \int_0^1 \left(x^2\right)\left(\frac{x}{\sqrt{1 - x^2}}\right)dx$$

and then applying integration by parts, determine its value.

[6 marks]

Question 6

(i) Given that $S = \int \frac{(\ln x)^k}{x}\, dx$ prove that $S = \frac{(\ln x)^{k+1}}{k+1} + c$ where k is a positive integer and c is the constant of integration

[5 marks]

(ii) Hence show that both $\int_0^1 \frac{(\ln x)^k}{x}\, dx$ and $\int_1^\infty \frac{(\ln x)^k}{x}\, dx$ are divergent where k is a positive integer.

[6 marks]