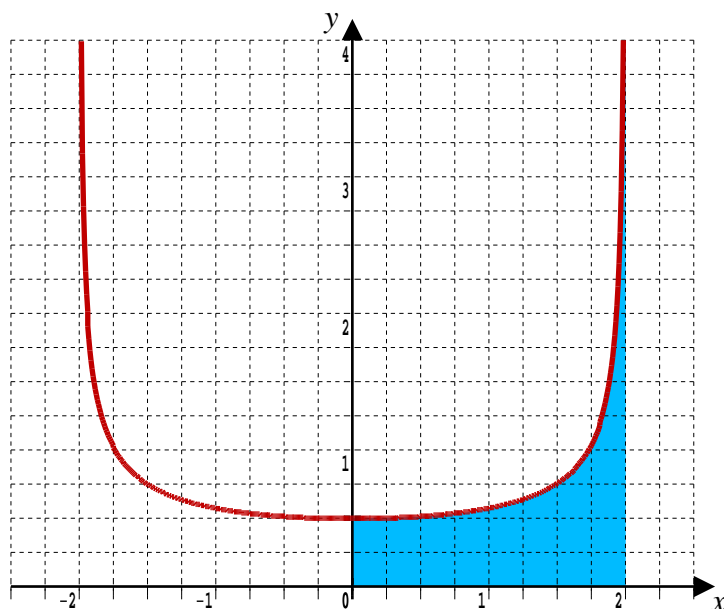


Lesson 3

Further A-Level Pure Mathematics, Core 2 Improper Integrals

3.1 Substitution

The graph is of the function $f(x) = \frac{1}{\sqrt{4-x^2}}$ and the task is to find the area shown shaded from $x=0$ to the vertical asymptote at $x=2$



Teaching video : <http://www.NumberWonder.co.uk/v9100/3.mp4>



[7 marks]

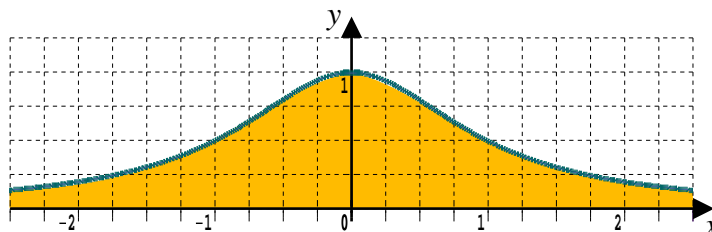
3.2 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 50

Question 1

The graph is of the function $f(x) = \frac{1}{1+x^2}$ and the task to find $\int_{-\infty}^{\infty} f(x) dx$



- (i) The function is even. Explain how this fact can simplify the task.

[1 mark]

- (ii) Prove that the derivative with respect to θ of $\tan \theta$ is $\sec^2 \theta$

[2 marks]

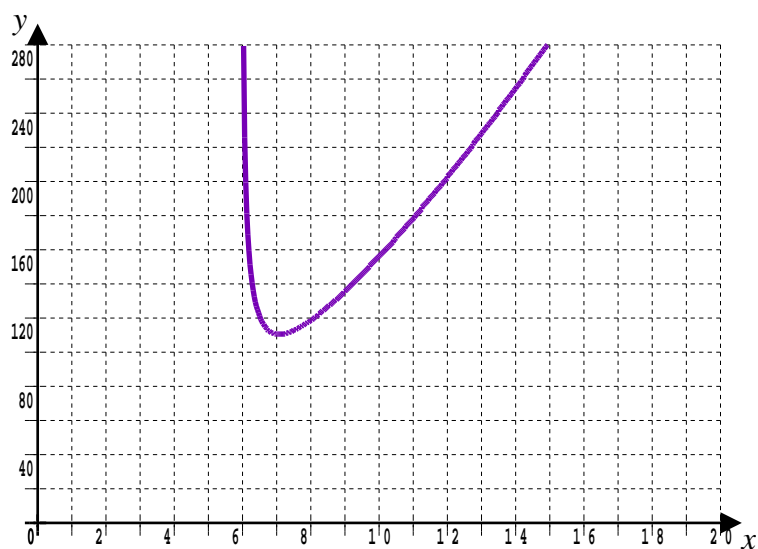
- (iii) Use the substitution $x = \tan \theta$ to show that $\int_{-\infty}^{\infty} f(x) dx = \pi$

A sketch of the graph of $\theta = \arctan x$ may help with changing the limits.

[6 marks]

Question 2

The graph is of $y = \frac{(x - 4)(5x + 2)}{\sqrt{x - 6}}$



- (i) Use the substitution $x = 6 + u^2$ to determine the value of the improper integral $\int_6^7 \frac{(x - 4)(5x + 2)}{\sqrt{x - 6}} dx$

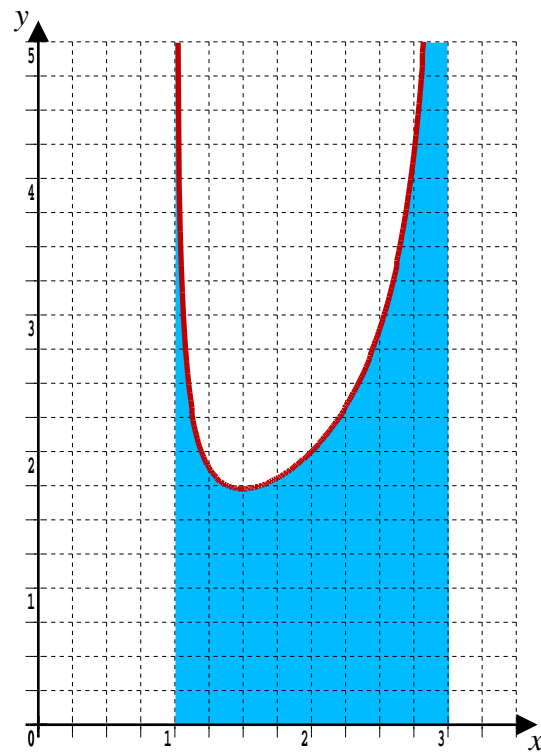
[5 marks]

- (ii) From the graph, explain why $\int_7^\infty \frac{(x - 4)(5x + 2)}{\sqrt{x - 6}} dx$ will be divergent

[2 marks]

Question 3

The graph is of the function $f(x) = \frac{x}{\sqrt{(x-1)(3-x)}}$

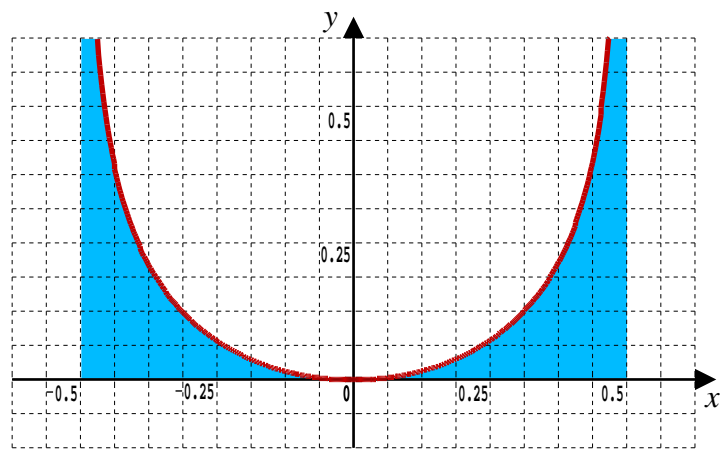


Use the substitution $x = \sin \theta + 2$ to determine the area shaded which is between the curve, the x -axis and the vertical asymptotes at $x = 1$ and $x = 3$

[7 marks]

Question 4

The graph is of the function $f(x) = \frac{x^2}{\sqrt{1-4x^2}}$



Use the substitution $x = \frac{1}{2}\sin \theta$ to determine the area shaded which is between the curve, the x -axis and the vertical asymptotes at $x = \pm \frac{1}{2}$

[7 marks]

Question 5

Faced with an improper integral of the form,

$$\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}} dx \quad \text{where } a \text{ and } b \text{ are constants with } a < b$$

the substitution to use is $x = a \cos^2 \theta + b \sin^2 \theta$

(i) Show that $(b-x)(x-a) = (b-a)^2 \cos^2 \theta \sin^2 \theta$

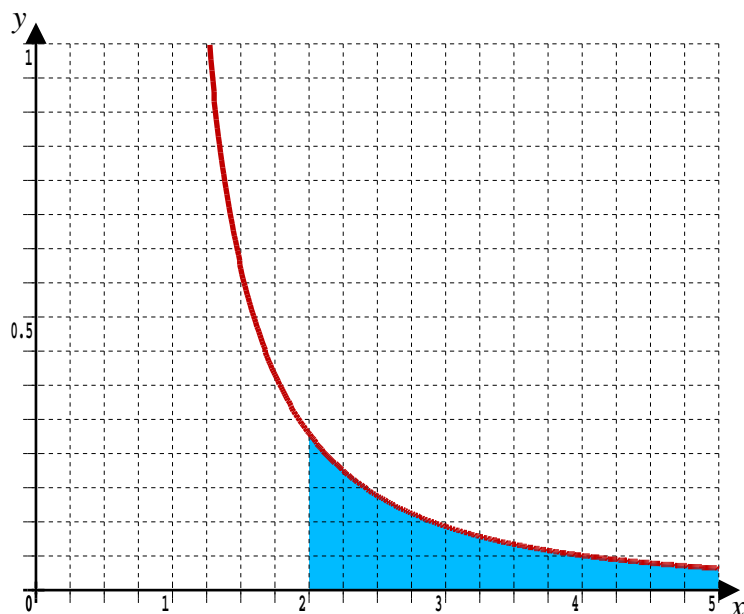
[3 marks]

(ii) Show that $\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}} dx = \pi$

[8 marks]

Question 6

The graph is of the function $f(x) = \frac{1}{x\sqrt{x^2 - 1}}$



Use the substitution $x = \sec \theta$ to determine the area between the curve, the x -axis, the vertical asymptote at $x = 2$ and extending indefinitely rightward as $x \rightarrow \infty$

[9 marks]

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In October 2020, Shrewsbury School was voted “**Independent School of the Year 2020**”

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Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk