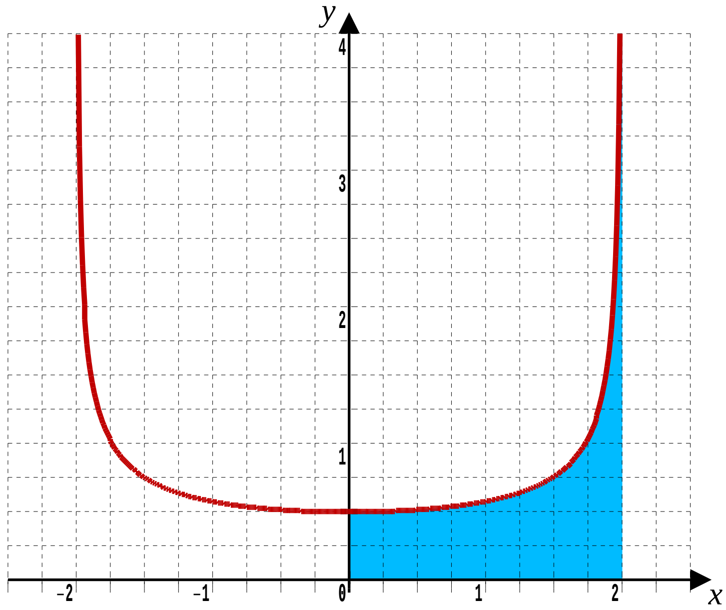
Lesson 3

Further A-Level Pure Mathematics, Core 2

Improper Integrals

3.1 Substitution

The graph is of the function $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$ and the task is to find the area shown shaded from $x = 0$ to the vertical asymptote at $x = 2$



Teaching video : http://www.NumberWonder.co.uk/v9100/3.mp4

[7 marks]

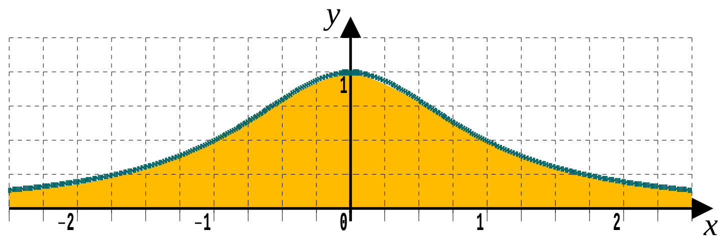
3.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

The graph is of the function $f(x) = \dfrac{1}{1 + x^2}$ and the task to find $\int_{-\infty}^{\infty} f(x)\ dx$



(i) The function is even. Explain how this fact can simplify the task.

[1 mark]

(ii) Prove that the derivative with respect to θ of $\tan\theta$ is $\sec^2\theta$

[2 marks]

(iii) Use the substitution $x = \tan\theta$ to show that $\int_{-\infty}^{\infty} f(x)\ dx = \pi$

A sketch of the graph of $\theta = \arctan x$ may help with changing the limits.

[6 marks]

Question 2

The graph is of $y = \dfrac{(x - 4)(5x + 2)}{\sqrt{x - 6}}$

(i) Use the substitution $x = 6 + u^2$ to determine the value of the improper integral $\displaystyle\int_6^7 \frac{(x - 4)(5x + 2)}{\sqrt{x - 6}}\, dx$

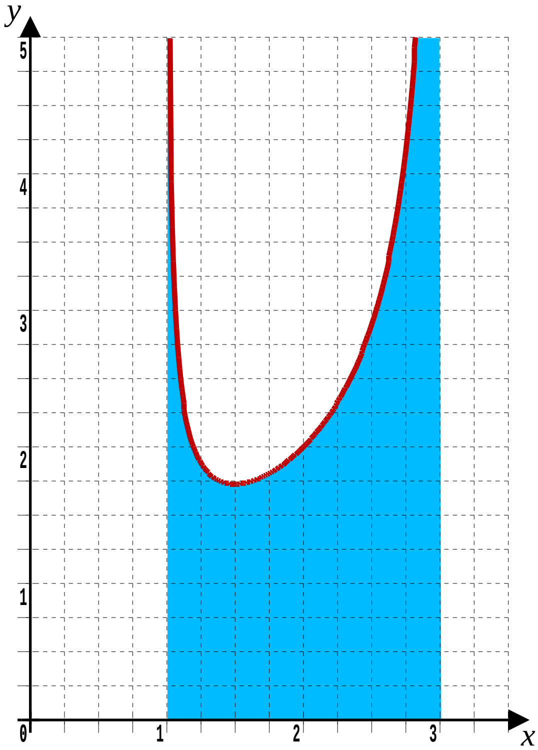
[5 marks]

(ii) From the graph, explain why $\displaystyle\int_7^\infty \frac{(x - 4)(5x + 2)}{\sqrt{x - 6}}\, dx$ will be divergent

[2 marks]

Question 3

The graph is of the function $f(x) = \dfrac{x}{\sqrt{(x-1)(3-x)}}$

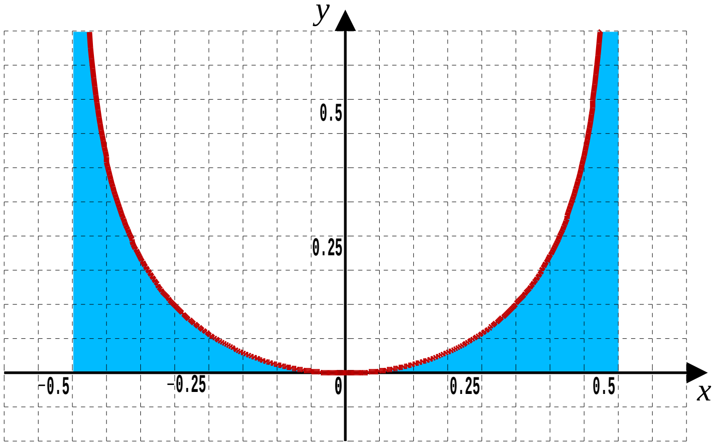


Use the substitution $x = \sin\theta + 2$ to determine the area shaded which is between the curve, the x-axis and the vertical asymptotes at $x = 1$ and $x = 3$

[7 marks]

Question 4

The graph is of the function $f(x) = \dfrac{x^2}{\sqrt{1 - 4x^2}}$



Use the substitution $x = \dfrac{1}{2} \sin \theta$ to determine the area shaded which is between the curve, the x-axis and the vertical asymptotes at $x = \pm \dfrac{1}{2}$

[7 marks]

Question 5

Faced with an improper integral of the form,

$$\int_a^b \frac{1}{\sqrt{(b - x)(x - a)}}\, dx \qquad \text{where } a \text{ and } b \text{ are constants with } a < b$$

the substitution to use is $x = a\cos^2\theta + b\sin^2\theta$

(i) Show that $(b - x)(x - a) = (b - a)^2 \cos^2\theta \sin^2\theta$

[3 marks]

(ii) Show that $\displaystyle\int_a^b \frac{1}{\sqrt{(b - x)(x - a)}}\, dx = \pi$

[8 marks]

Question 6

The graph is of the function $f(x) = \dfrac{1}{x\sqrt{x^2 - 1}}$

Use the substitution $x = sec\,\theta$ to determine the area between the curve, the x-axis, the vertical asymptote at $x = 2$ and extending indefinitely rightward as $x \to \infty$

[9 marks]