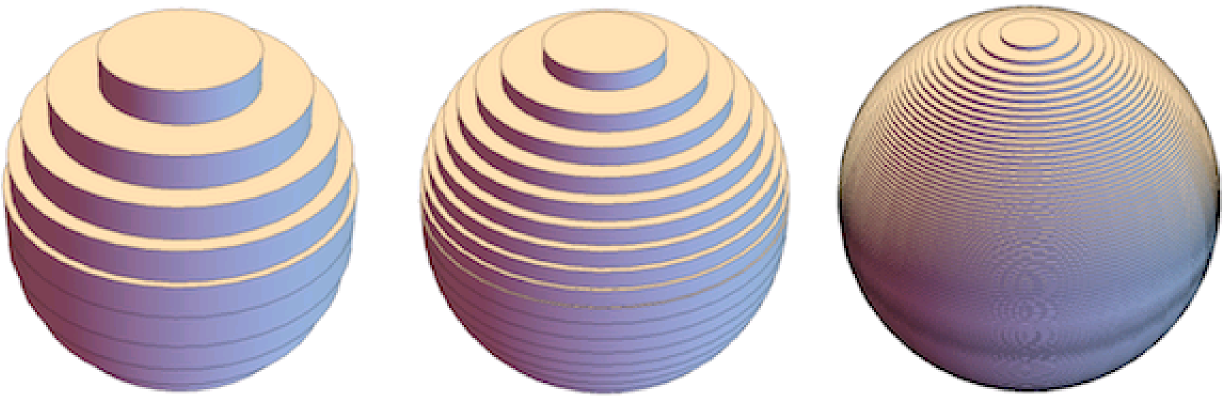
5.1 Volume of Revolution

If a solid, when sliced, has cross sections that are circular then its volume may be found using the idea of a profile curve being spun about either the x or the y-axis. This is called the "discs" method of finding a volume as the integration is finding the sum of a number cylinders, as that number tends to infinity, and as the thickness of the discs involved correspondingly tends to zero. In the limit, the volume found is exact.



As the number of discs in increased the volume of this sphere is better approximated by the sum of the volumes of those discs
From left to right: 10 discs, 20 discs and 100 discs
Image made in Wolfram Alpha by Martin Hansen

In the A-Level Further Mathematics course this is the only method taught but there other techniques such as the "shell" method or the "slab" method that can tackle a variety of other shapes; They are easily learnt should the need arise.

For the "discs" method the key result is the following;

The Volume of Revolution Formulae

- When $y = f(x)$ is rotated $2\pi^c$ about the x-axis on interval $a \leqslant x \leqslant b$

$$Volume = \pi \int_a^b y^2 \, dx$$

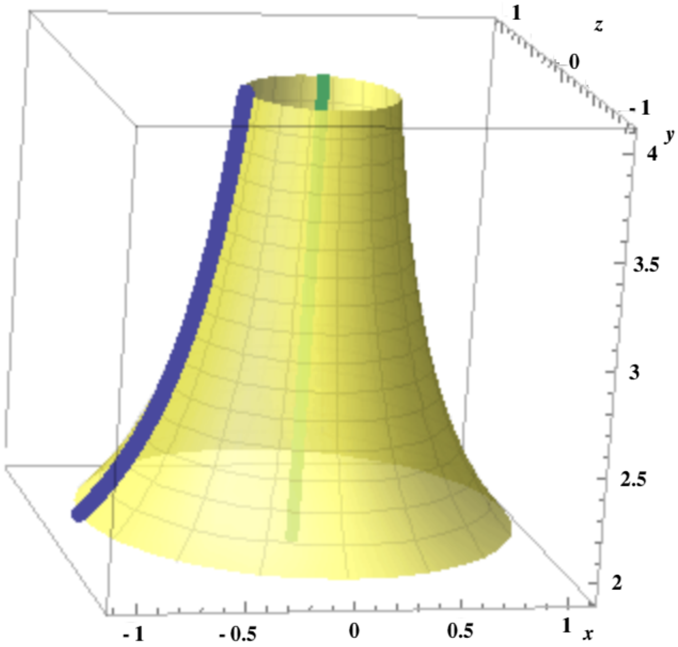
- When $x = f(y)$ is rotated $2\pi^c$ about the y-axis on interval $a \leqslant y \leqslant b$

$$Volume = \pi \int_a^b x^2 \, dy$$

In this lesson the focus is upon questions that involve improper integrals.
Any of the integration techniques studied previously may be called upon.

5.2 Example

Find the volume swept out when the profile curve $x = \dfrac{1}{\sqrt{y}\ \ln y}$ is rotated $2\pi^c$ about the y-axis, $2 \leqslant y < \infty$. The lower end of the solid is depicted below.



Teaching Video : http://www.NumberWonder.co.uk/v9100/5.mp4

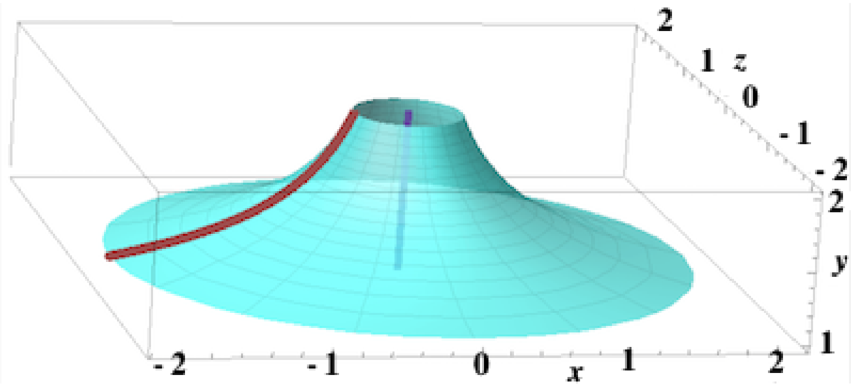
[5 marks]

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Find the volume swept out when the profile curve $x = \dfrac{y + 1}{y^3}$ is rotated $2\pi^c$ about the y-axis, $1 \leqslant y < \infty$. The lower end of the solid is depicted below.



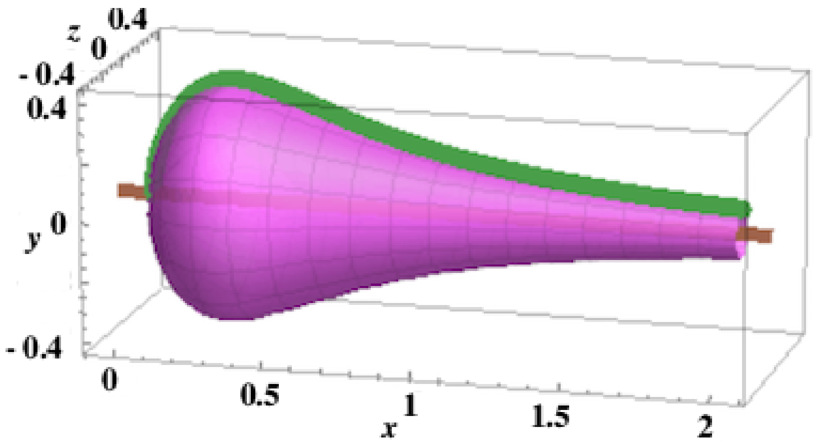
[5 marks]

Question 2

Find the volume swept out when the profile curve $y = \dfrac{\sqrt{x}}{1 + 4x^2}$ is rotated $2\pi^c$ about the x-axis, $0 \leqslant x < \infty$.

The integration involved is a "chain rule backwards".

The lower end of the solid is depicted below.

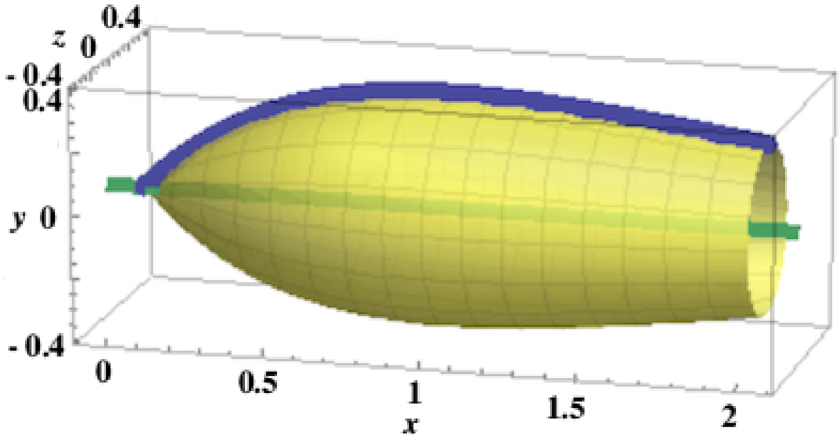


[5 marks]

Question 3

Find the volume swept out when the profile curve $y = x\,e^{-x}$ is rotated $2\pi^c$ about the x-axis, $0 \leqslant x < \infty$. The lower end of the solid is depicted below.

You may use without proof the fact that $\underset{t\to\infty}{limit}\ t^n e^{-2t} = 0$ for $n \in \mathbb{Z}^+$



[7 marks]

Question 4

(i) By means of a suitable trigonometric substitution, which you should clearly state, determine the value of the improper integral,

$$\int_0^{\infty} \frac{1}{x^2 + 9} \, dx$$

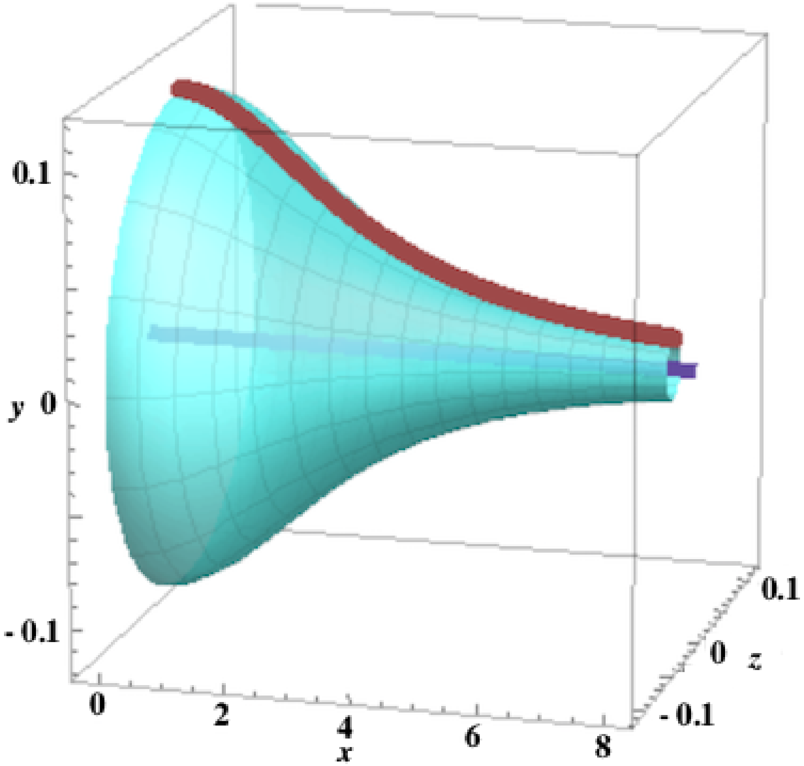
[4 marks]

(ii) By thinking of $\displaystyle\int \frac{1}{x^2 + 9} \, dx$ as $\displaystyle\int \left(\frac{1}{x^2 + 9} \right) (1) \, dx$ and using integration by parts show that,

$$\int \frac{1}{x^2 + 9} \, dx = \frac{x}{x^2 + 9} + 2 \int \frac{1}{x^2 + 9} \, dx - 18 \int \frac{1}{(x^2 + 9)^2} \, dx$$

[4 marks]

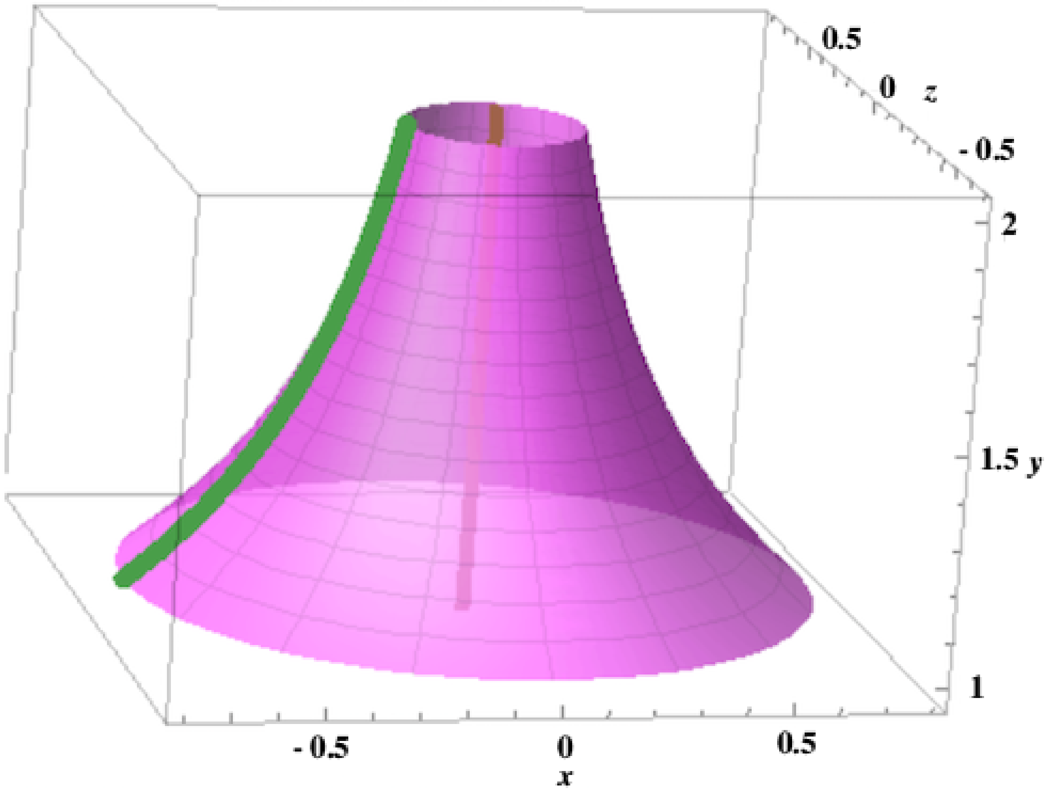
(iii) Find the volume swept out when the profile curve $y = \dfrac{1}{x^2 + 9}$ is rotated by $2\pi^c$ about the x-axis, $0 \leqslant x < \infty$.



[4 marks]

Question 5

Find the volume swept out when the profile curve $x = \dfrac{\sqrt{4y + 1}}{y(y + 1)}$ is rotated $2\pi^c$ about the y-axis, $1 \leqslant x < \infty$. The lower end of the solid is depicted below.



[11 marks]