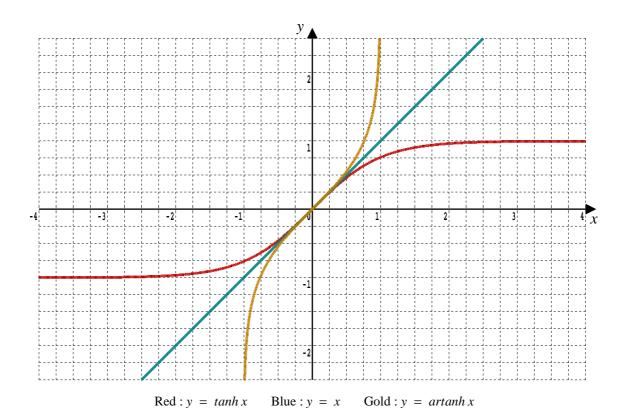
7.1 artanh x

Like $sinh\ x$, the function $tanh\ x$ is one-to-one on an unrestricted domain although, unlike $sinh\ x$ it has horizontal asymptotes at $y=\pm 1$. Like any one-to-one function, it has an inverse that, graphically, is a reflection in y=x. The inverse of $tanh\ x$ is called $artanh\ x$.



Like sinh x and cosh x, the tanh x function is defined in terms of exponentials, and its inverse, artanh x, involves logarithms.

The Inverse Of tanh x : artanh x

$$artanh x = \frac{1}{2} ln \left(\frac{1+x}{1-x} \right) \qquad x \in \mathbb{R}, |x| < 1$$

A proof of this is written out on the next page. An excellent video from *Exam Solutions* talks through the proof



7.2 The Proof

$$y = \operatorname{artanh} x$$

$$\therefore x = \operatorname{tanh} y$$

$$= \frac{\sinh y}{\cosh y}$$

$$= \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} \times \frac{e^{y}}{e^{y}}$$

$$= \frac{(e^{y})^{2} - 1}{(e^{y})^{2} + 1}$$

$$x(e^{y})^{2} + x = (e^{y})^{2} - 1$$

$$1 + x = (e^{y})^{2} - x(e^{y})^{2}$$

$$1 + x = (e^{y})^{2} (1 - x)$$

$$e^{y} = \pm \sqrt{\frac{1 + x}{1 - x}}$$
Now, since $e^{y} > 0$, $e^{y} = \sqrt{\frac{1 + x}{1 - x}}$

$$= \left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}}$$

$$y = \ln\left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}}$$
That is, $\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$

The domain of the inverse function is the range of the original function

That is
$$x \in \mathbb{R}$$
, $|x| < 1$

7.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 35

Question 1

Further A-Level Examination Question from June 2018, FP3, Q1 (Edexcel) Solve the equation

$$15 \operatorname{sech}^2 x + 7 \tanh x = 13$$

Give your answers in terms of simplified natural logarithms

Question 2

Further A-Level Examination Question from June 2009, FP3, Q1 (Edexcel) Solve the equation

$$7 \operatorname{sech} x - \tanh x = 5$$

Give your answers in the form ln a where a is a rational number

[5 marks]

Question 3

Find the equation of the tangent at the point where $x = \frac{12}{13}$ on the curve with equation $y = \operatorname{artanh} x$

Question 4

Further A-Level Examination Question from June 2006, FP2, Q5 (Edexcel) The curve with equation

$$y = -x + \tanh(4x) \qquad x \ge 0$$

has a maximum turning point A

(a) Find, in exact logarithmic form, the x-coordinate of A

[4 marks]

(**b**) Show that the y-coordinate of A is $\frac{1}{4} \{ 2\sqrt{3} - ln(2 + \sqrt{3}) \}$

Question 5

Given that,

$$artanh x + artanh y = ln \sqrt{3}$$

prove that

$$y = \frac{2x - 1}{x - 2}$$



Further A-Level Examination Question from June 2008, FP2, Q1 (Edexcel)

Show that
$$\frac{d}{dx} [ln(tanh x)] = 2 csch(2x)$$
 $x > 0$

[4 marks]

Question 7

Further A-Level Examination Question from June 2004, P5, Q1(b) (Edexcel) Solve $\operatorname{csch} x - 2 \operatorname{coth} x = 2$ giving your answer in the form $\operatorname{kln} a$ where $\operatorname{kln} a$ are integers

[4 marks]