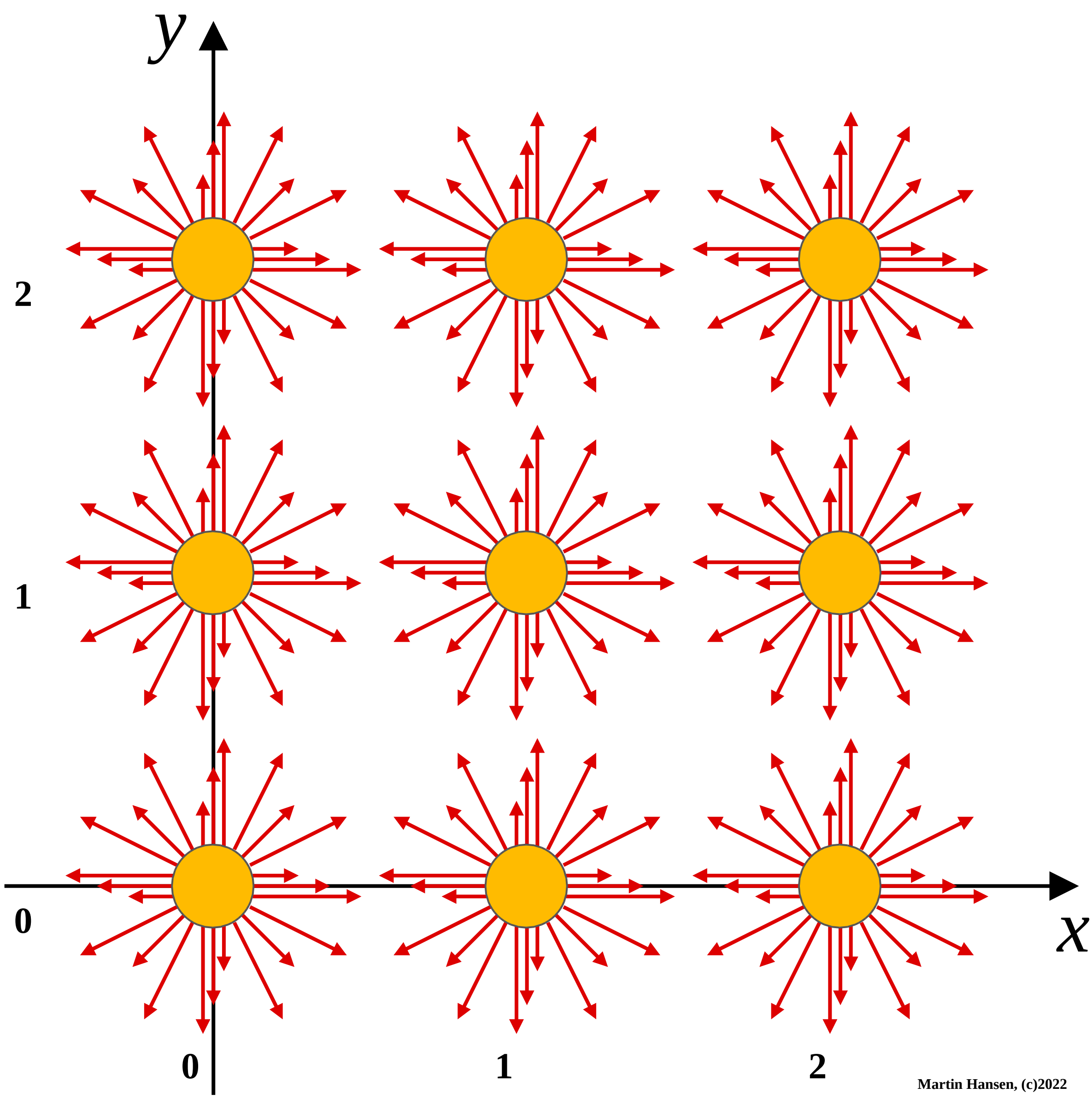
University Undergraduate Lectures in Mathematics
A Second Year Course

VECTOR SPACES



An illustration of the vector space $\mathbb{F}_3$

V E C T O R   S P A C E S

Lecture 1

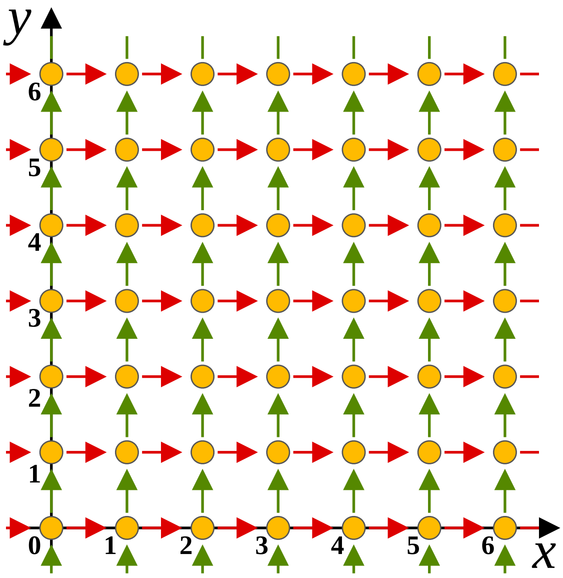
University Undergraduate Lectures in Mathematics
A Second Year Course
Vector Spaces

1.1 What is a Vector Space ?

A vector space consists of vectors and an arithmetic that acts upon those vectors. It's specified over a field using a minimal set of vectors termed a basis. For example, the set of vectors over the field of "integer 2-dimensional coordinates modulo seven" could be the definition of a particular vector space.

A suitable basis is $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and the points within the vector space are obtained from all possible linear combinations, modulo seven, of the basis vectors.

Colouring $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ red and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ green yields this visualisation:



There is an exploited ambiguity between thinking of a point as a point, or as a displacement vector (from the origin to the point).

The included points are derived from all possible values of $a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where a and b are modulo seven integers with additions and multiplications modulo seven.

In fact, given a couple of points within any vector space, $(\, u,\, v\,)$ and $(\, x,\, y\,)$, all calculations of the type, $a\begin{pmatrix} u \\ v \end{pmatrix} + b\begin{pmatrix} x \\ y \end{pmatrix}$ must yield an answer in the vector space.

This is termed "closure". In general, the arithmetic acting on the vector space is not specified separately; it's intimately tied to the vector space itself. Roughly speaking, "the normal rules of arithmetic apply". The technical detail of this will be given once some experience of working with vector spaces has been acquired.

1.2 What is a Suitable Basis ?

Let $\mathbb{Z}_7^2$ be the two-dimensional vector space of modulo seven coordinate points.
A basis has to comprise of a minimal set of linearly independent vectors.
That is, for a basis of two vectors, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, there must be no non-zero integer constant k for which $\boldsymbol{v}_1 = k\,\boldsymbol{v}_2$

As observed previously, a suitable basis is $B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ because you can get to any point in the vector space in a unique way via a linear combination of those two basis vectors. For example, to get to the point $(1,\ 4)$,

$$1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 4 \end{pmatrix} \pmod 7$$

1.2.1 Problem

Show that another basis for $\mathbb{Z}_7^2$ is $B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ by first exhibiting a linear combination of those two basis vectors that gets to the point $(1,\ 4)$ and then a linear combination that gets to the generalised point (x, y).

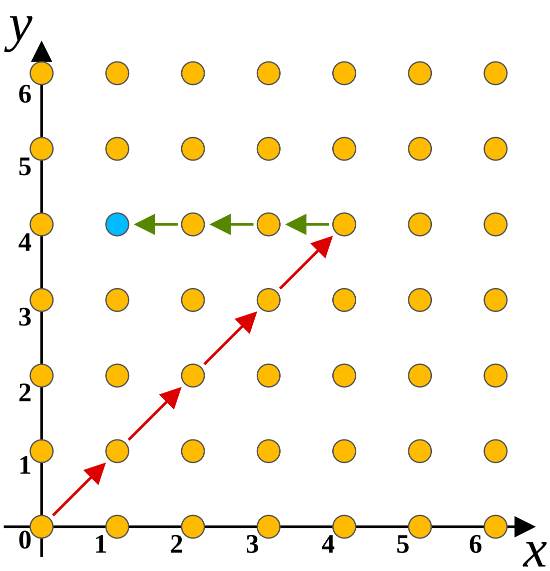
[6 marks]

1.2.2 Solution

It's possible to guess the answer but here's a methodical method.

We require, $a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 4 \end{pmatrix} \pmod 7 \Rightarrow \begin{cases} a - b \equiv 1 \pmod 7 \\ a \equiv 4 \pmod 7 \end{cases}$

Solve these equations simultaneously to get $a \equiv 4 \pmod 7$ and $b \equiv 3 \pmod 7$



An illustration of the solution $4\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 4 \end{pmatrix} \pmod 7$

In general, $a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} x \\ y \end{pmatrix} \pmod 7 \Rightarrow \begin{cases} a - b \equiv x \pmod 7 \\ a \equiv y \pmod 7 \end{cases}$

Solving the two resulting equations simultaneously this time yields that,

$$a \equiv y \pmod 7 \text{ and } b \equiv y - x \pmod 7$$

This shows that any point in the vector space can be reached.
Notice that for integer values of *x* and *y*, these give integer values of *a* and *b*.

$$\therefore \; B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\} \text{ is a suitable basis.} \qquad \square$$

[6 marks]

1.3 A Polynomial Example

Let $\mathbb{F}_5$ be the field of integers modulo five.

Let *V* be the vector space of polynomials of the form,

$$p(x) \equiv a + bx + cx^2 \text{ where } a, b, c \in \mathbb{F}_5$$

(a) Write down,

(i) the scalar multiple $3(2 + 3x + 4x^2)$

(ii) the sum of $1 + 3x + 4x^2$ and $3 + 2x + 4x^2$

(iii) the zero vector in *V*

(iv) the inverse of the vector $2 + 3x + 4x^2$

Given that a suitable basis for *V* is $B = \{1 + x^2, \; x, \; x + x^2\}$

(b) Express $4 + x + 3x^2$ using *B*

Teaching Video: http://www.NumberWonder.co.uk/v9112/1a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9112/1b.mp4 (Part 2)

<= Part 1

Part 2 =>

[8 marks]

1.4 A Proof Example

In this section a proof will be developed to answer the following question;

> For two bases of a vector space, one is a subset of the other.
> Show that these bases are identical.

[8 marks]

1.4.1 Preliminary Thoughts #1

Suppose that $B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

If we try to take $B_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ as our subset basis then we can only move along the x-axis and get to integer points on that axis; B_3 does not span the entire space.

There are points in the vector space, such as $(1,\ 1)$ that we can't get to.

With reluctance it has to be accepted that B_3 has to have at least one other vector.

However, B_3 has to be a subset of B_1 : we're forced to extend B_3 to be identical to B_1.

1.4.2 Preliminary Thoughts #2

The same situation would arise if we started with $B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ and tried for a subset basis of, say, $B_4 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. We could only move along and get to the integer points on the line with equation, $y = x$ and so B_4 does not span the space.

Once again there are points in the vector space, such as $(1,\ 4)$ that we can't get to.

To be able to move in the other dimension another vector has to be added to the basis that is not parallel to the existing vector. In order to extend B_4 whilst keeping it a subset of B_2 the only possible extension results in B_4 being identical to B_2.

1.4.3 The Proof Technique

We are now set up to make the leap to a generalised proof to the question.

The proof technique to be used is to propose that two different versions of a “thing” exist but, via logical deduction, be forced to conclude that the proposed two different “things” have to actually be identical.

1.4.4 The Proof

Let a basis that spans an n-dimensional vector space, L^n, be,

$$L_1 = \{ \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots, \boldsymbol{v}_n \}$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots, \boldsymbol{v}_n$ are a set of linearly independent vectors.

Consider now a basis, L_2, that also spans L^n and which is a subset of L_1

Let us suppose that $L_2 = \{ \boldsymbol{v}_1 \}$. This proposal fails to span L^n because there are points in L^n of the form $a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ that can not be reached by $k\boldsymbol{v}_1$ where a, b and k are integer constants.

In other words, there is no solution to the equation,

$$a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = k\boldsymbol{v}_1$$

We are thus forced to add the vector $\boldsymbol{v}_2$ to our proposed basis, L_2 which will then be $L_2 = \{ \boldsymbol{v}_1, \boldsymbol{v}_2 \}$. The logic now repeats with the observation that there are still points in L_2 that cannot be reached, this time of the form $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3$

L_2 still does not span L^n as there is no solution to the equation,

$a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3 = k\boldsymbol{v}_1 + l\boldsymbol{v}_2$ where a, b, c, k and l are integer constants.

This inductive argument repeats until we have been forced to add all the basis vectors of L_1 to L_2.

$$L_1 = L_2 = \{ \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots, \boldsymbol{v}_n \}$$

So, given one basis of a linear space, in trying to create another basis that is different to the first but also a subset of the first we have been forced to conclude that the two bases are, in fact, identical. □

[8 marks]

1.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Tony proposes the following basis for the vector space $\mathbb{Z}_p^3$ where p is prime.

$$B = \left\{ \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

However, this set of vectors are not linearly independent.

Show this by finding integer values of a and b such that $a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 = \boldsymbol{v}_3$

[2 marks]

Question 2

Fred proposes the following basis for the vector space $\mathbb{Z}_p^4$ where p is prime.

$$B = \left\{ \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -4 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -3 \\ 6 \\ -4 \\ 1 \end{pmatrix} \right\}$$

However, this set of vectors are not linearly independent.

Show this by finding integer values of a and b such that $a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 = \boldsymbol{v}_3$

[2 marks]

Question 3

Show that the basis $B_3 = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$ is a suitable basis for $\mathbb{Z}_7^2$

Do this by first exhibiting a linear combination of the two basis vectors that gets to the point $(1, \ 1)$ and then a linear combination that gets to the point $(x, \ y)$.

[6 marks]

Question 4

Walter proposes $B = \left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ as a basis for $\mathbb{Z}_p^3$

Show that this is a valid proposal.

What are the coordinates of a general vector $\boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ relative to the basis vectors ?

[6 marks]

Question 5

Let $\mathbb{F}_7$ be the field of integers modulo seven.

Let V be the vector space of polynomials of the form,

$$p(x) \equiv a + bx + cx^2 \text{ where } a, b, c \in \mathbb{F}_7$$

(a) Write down,

(i) the scalar multiple $6\left(5 + 4x + 6x^2\right)$

[1 mark]

(ii) the sum of $4 + 6x^2$ and $3 + 2x + 4x^2$

[1 mark]

(iii) the zero vector in V

[1 mark]

(iv) the inverse of the vector $5 + 2x + x^2$

[1 mark]

Given that a suitable basis for V is $B = \left\{1,\ 4x,\ 4x + x^2\right\}$

(b) express $3 + x + 5x^2$ using B

[4 marks]

Question 6

Let $V = \mathbb{P}_4$ be the vector space of polynomials of degree less than 4 over the field $\mathbb{R}$.

A general polynomial within this vector space is then

$$p(x) = ax^3 + bx^2 + cx + d \quad \text{for } a, b, c, d \in \mathbb{R}$$

Let W be the vector subspace of polynomials $p(x)$ such that $p(0) = p(1) = 0$

(i) Show that $p(x) = ax^3 + bx^2 + (-a - b)x$

[4 marks]

(ii) Hence, find a basis for W and state its dimension.

[3 marks]

(iii) Extend the basis for W to be a basis for V and state its dimension.

[2 marks]

Question 7
Consider the set A of all arithmetic progressions.
These form a vector space over the field $\mathbb{R}$.
Verify that, under all possible linear combinations of addition and scalar multiplication, the set A is closed.

[5 marks]

Question 8
Consider the set G of all geometric progressions.
These do not form a vector space over the field $\mathbb{R}$.
Show this by demonstrating that not all possible combinations of addition and scalar multiplication give a closed system.

[4 marks]

Question 9

Let U be the vector subspace of $\mathbb{R}^5$ defined by,

$$U = \left\{ \left(x_1,\ x_2,\ x_3,\ x_4,\ x_5 \right) \in \mathbb{R}^5 : x_1 = 3x_2 \ \text{ and } \ x_3 = 7x_4 \right\}$$

Find a basis of U

[4 marks]

Question 10

The basis $B = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ defines a vector subspace, a plane, within $\mathbb{R}^3$

Determine the Cartesian equation of the plane.

[4 marks]