Lesson 2

A-Level Pure Mathematics : Year 2
Integration III

2.1 Integration by Parts

This is a product rule for integration.
Use it when a product is to be integrated, for example;

$$\int x \ \sin x \ dx$$

Integration by parts expands such integrals into four pieces. It requires some differentiation, ***D***, as well as integration, ***I***, and some leaving alone, ***L***.

The mnemonic ***L I D I*** may help. (*Lie -Die*)

In the examples we'll be integrating and differentiating $\sin x$, $\cos x$, $\ln x$ and e^x
Here is a reminder of their derivatives:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Example N° 1

$$\int x \ \sin x \ dx$$

$$= \ \boldsymbol{L}(x) \quad \boldsymbol{I}(\sin x) - \int \boldsymbol{D}(x) \quad \boldsymbol{I}(\sin x) \ dx$$

$$= \ x \quad (-\cos x) - \int 1 \quad (-\cos x) \ dx$$

$$= -x \ \cos x + \int \cos x \ dx$$

$$= -x \ \cos x + \sin x + c$$

[3 marks]

Example N° 2

$$\int x^3 \; ln\,x \; dx$$

First, swap the order

as we can $\boldsymbol{D}(ln\,x)$ but not (yet!) $\boldsymbol{I}(ln\,x)$

$$= \int ln\,x \; x^3 \; dx$$

$$= \; \boldsymbol{L}(ln\,x) \;\; \boldsymbol{I}\left(x^3\right) - \int \; \boldsymbol{D}(ln\,x) \;\; \boldsymbol{I}\left(x^3\right) \; dx$$

$$= \quad ln\,x \quad \frac{x^4}{4} \; - \; \int \; \frac{1}{x} \quad \frac{x^4}{4} \quad dx$$

$$= \frac{x^4\, ln\,x}{4} - \int \frac{x^3}{4} \; dx$$

$$= \frac{x^4\, ln\,x}{4} - \frac{x^4}{16} + c$$

[3 marks]

Example N° 3

$$\int ln\,x \; dx$$

Sneaky question because this

does not look like a product

$$= \int ln\,x \;\; \times \; 1 \; dx$$

$$= \; \boldsymbol{L}(ln\,x) \;\; \boldsymbol{I}(1) - \int \; \boldsymbol{D}(ln\,x) \;\; \boldsymbol{I}(1) \; dx$$

$$= \quad ln\,x \quad\; x \; - \; \int \; \frac{1}{x} \quad x \quad dx$$

$$= x\, ln\,x - \int 1 \; dx$$

$$= x\, ln\,x - x + c$$

A mystery is solved; *ln x* can be integrated as well as differentiated !

[3 marks]

2.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Find the integral;

$$\int x \ \cos x \ dx$$

[3 marks]

Question 2

Find the integral;

$$\int x \ \sin 4x \ dx$$

[4 marks]

Question 3

This is the only question in the exercise were it's necessary to swap the order of the product in order to determine the integral, like Example N° 2

$$\int x^5 \ \ln x \ dx$$

[3 marks]

Question 4
Find the integral;

$$\int x\ e^{x}\ dx$$

[3 marks]

Question 5
Find the integral;

$$\int x\ e^{-5x}\ dx$$

[4 marks]

Question 6
Find the integral;

$$\int \frac{x}{2\,e^{x}}\ dx$$

[4 marks]

Question 7

(i) By setting up a chain rule backwards, find

$$\int (3x + 1)^6 \, dx$$

[2 marks]

(ii) Use your part (i) answer and integration by parts to show that

$$\int x (3x + 1)^6 \, dx = \frac{x (3x + 1)^7}{21} - \frac{(3x + 1)^8}{21 \times 24} + c$$

[4 marks]

(iii) Simplify your answer by showing that

$$\frac{x (3x + 1)^7}{21} - \frac{(3x + 1)^8}{21 \times 24} + c = \frac{1}{504} (3x + 1)^7 (21x - 1) + c$$

[3 marks]

Question 8

Evaluate, giving an exact answer,

$$\int_0^{\frac{\pi}{3}} x \ \sin 3x \ dx$$

[6 marks]

Question 9

Evaluate giving an exact answer,

$$\int_0^1 (2x + 1) \, e^x \ dx$$

[6 marks]

Question 10

Do not use integration by parts at any stage in this question !

Instead, use the substitution $u = 3x - 5$ to evaluate,

$$7 \int_1^2 x^2 \, (3x - 5)^4 \, dx$$

[8 marks]