Lesson 3

A-Level Pure Mathematics : Year 2

Integration III

3.1 Tricky Integration by Parts

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Using the table of derivatives from right to left we can see that, for example;

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

However, there are some obvious omissions.
For example;

$$\int \tan x \, dx$$

Some cunning is needed to find this integral.

$$\begin{aligned}
\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\
&= \int (\sin x)(\cos x)^{-1} \, dx \\
&= (-1) \int (-\sin x)(\cos x)^{-1} \, dx \\
&= (-1) \ln |\cos x| + c \\
&= \ln |\cos x|^{-1} + c \\
&= \ln \left| \frac{1}{\cos x} \right| + c \\
&= \ln |\sec x| + c
\end{aligned}$$

[4 marks]

3.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Use the result, just proved, to find show that;

$$\int_0^{\frac{\pi}{6}} tan\, x \;\; dx \;=\; ln\, 2 \;-\; \frac{1}{2}\, ln\, 3$$

[4 marks]

Question 2

Use integration by parts to find;

$$\int \; x \;\; sec^2 x \;\; dx$$

[4 marks]

Question 3

(a) Use the fact that;

$$\cos^2 x + \sin^2 x = 1$$

to prove that;

$$\tan^2 x = \sec^2 x - 1$$

[2 marks]

(b) Hence, or otherwise, find;

$$\int x \tan^2 x \, dx$$

[3 marks]

Question 4

Mirror the "cunning" used to integrate $\tan x$ to find an expression for;

$$\int \cot x \, dx$$

[4 marks]

Question 5

Use your question 4 result to find the ***exact*** value of;

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} cot\, x \;\; dx$$

[4 marks]

Question 6

Use integration by parts to find;

$$\int \; x \;\; csc^2 x \;\; dx$$

[5 marks]

Question 7

Using a trigonometric formula and integration by parts, or otherwise, find;

$$\int x \cot^2 x \, dx$$

[5 marks]

Question 8

Using a trigonometric formulae first, or otherwise, find;

$$\int \cos^2 x \, dx$$

[5 marks]

Question 9

Use integration by parts, and your question 8 result, to find;

$$\int x \cos^2 x \, dx$$

[7 marks]

Question 10

Use integration by parts to help find;

$$\int \frac{\ln x}{x} \, dx$$

[7 marks]