Lesson 2

2.1 Divisibility

In mathematics, seemingly simple results can have far reaching consequences. One such result is the following.

The Sum Divisor Rule

Let x, y and n be three natural numbers.
If both $(x + y)$ and x are divisible by n then y must also be divisible by n.

Proof

Let the sum by x and y be z where z will be another natural number.

$$x + y = z$$

Both x and z are divisible by n so I can write that $x = na$ and $z = nb$ for some natural numbers a and b, giving,

$$na + y = nb$$
$$y = nb - na$$
$$= n(b - a)$$

which shows that y is divisible by n as the rule claims ☐

2.2 Divisibility by 2

It is easy to tell if a large number is even or odd simply by looking at the rightmost digit, R, of the number.

If $R \in \{0, 2, 4, 6, 8\}$ then the large number is even.

If $R \in \{1, 3, 5, 7, 9\}$ then the large number is odd.

But why does this test work ?

Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits.

Then, $$aR = 10a + R$$

By the Sum Divisor Rule, when aR and $10a$ are both divisible by 2, so must R be. In other words when $R \in \{0, 2, 4, 6 \text{ or } 8\}$, aR is even, otherwise odd.

Divisibility : How to count the number of sheep in a field.
Step 1 : Count the number of legs.
Step 2 : Divide by four.

2.3 Another Divisibility Test

The test for whether or not a number is divisible by 2 begs the question, "Are there similar, straightforward techniques to determine if a large number is divisible by other primes" ?

The answer is that there are, although they are not as well known.

Divisibility by 7

Subtract twice the last digit from the number formed by the remaining digits.

Repeat as necessary.

If, at any stage, a result is obtained that is obviously divisible by 7 then the original number is also divisible by 7

2.3.1 Example #1

Without using a calculator, show that 1603 is divisible by 7

[2 marks]

$1603 \Rightarrow 160 - 6 = 154$

$\Rightarrow 15 - 8 = 7$ which is obviously divisible by 7

$\therefore$ 1603 is divisible by 7

[2 marks]

2.3.2 Example #2

Without using a calculator, show that 504091 is divisible by 7

[2 marks]

2.3.3 Example #3

Provide a proof for the test for divisibility by 7

[7 marks]

2.4 Another Divisibility Rule

In the proof of the divisibility by 7 test, use was made of the Product Divisor Rule, which you may feel is obvious.

The Product Divisor Rule

Let x, y and n be three natural numbers.
If xy is divisible by n and x is not divisible by n then y must be divisible by n.

Proof

Let the product of x and y be z where z will be another natural number.

$$xy = z$$

As z is divisible by n, I can write that $z = na$ for some $a \in \mathbb{N}$.

$$xy = na$$

$$\frac{xy}{n} = a$$

As a cannot be a fraction, one of x or y must be writable as nb for some $b \in \mathbb{N}$ so that the n on the denominator can be cancelled with an n on the numerator.
But x is not divisible by n and so $x \neq nb$.
Therefore, $y = nb$ which shows y must be divisible by n as claimed. ☐

2.5 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40 marks

Question 1

(i) Show that 24094 is divisible by 7 using the divisibility by 7 test.

[2 marks]

(ii) Explain why, after part (a), you immediately know that 24094 is divisible by 14

[2 marks]

Question 2

Young schoolchildren often think, mistakenly, that 91 is a prime number.
Show that 91 is divisible by 7 using the divisibility by 7 test.

[1 mark]

Question 3

The four digit numbers 5775, 3223 and 6996 are all of the form *abba* where a and b are natural numbers between 0 and 9 inclusive.
(That is $a, b \in \mathbb{N},\ 0 \leqslant a \leqslant 9,\ 0 \leqslant b \leqslant 9$)
Prove that all such four digit numbers of the form *abba* are divisible by 11.

Hint : $abba = 1000a + 100b + 10b + a$

[4 marks]

Question 4

Prove that the three digit number *aba* when added to the three digit number *bab* is always divisible by 37

[5 marks]

Question 5

(i) Prove that for any four digit number *abcd* to be divisible by 3, the sum of its digits must be divisible by 3

[5 marks]

(ii) Hence explain how you know that 7458 is divisible by 3

[2 marks]

Question 6

(i) Prove that, for any natural number of more than two digits to be divisible by 4, the two rightmost digits must be divisible by 4

[6 marks]

(ii) Hence explain how you know that 290547452 is divisible by 4

[2 marks]

Question 7

Divisibility by 13
Add four times the last digit to the number formed by the remaining digits.
Repeat as necessary.
If at any stage a result is obtained that is obviously divisible by 13 then the original number is divisible by 13

(i) Without using a calculator, show that 119483 is divisible by 13

[2 marks]

(ii) Without using a calculator, show that 504088 is divisible by 13

[2 marks]

(iii) Provide a proof for the test for divisibility by 13

[7 marks]