Lesson 3

A-Level Pure Mathematics, Year 1

Proof I : The Art of Absolute Certainty

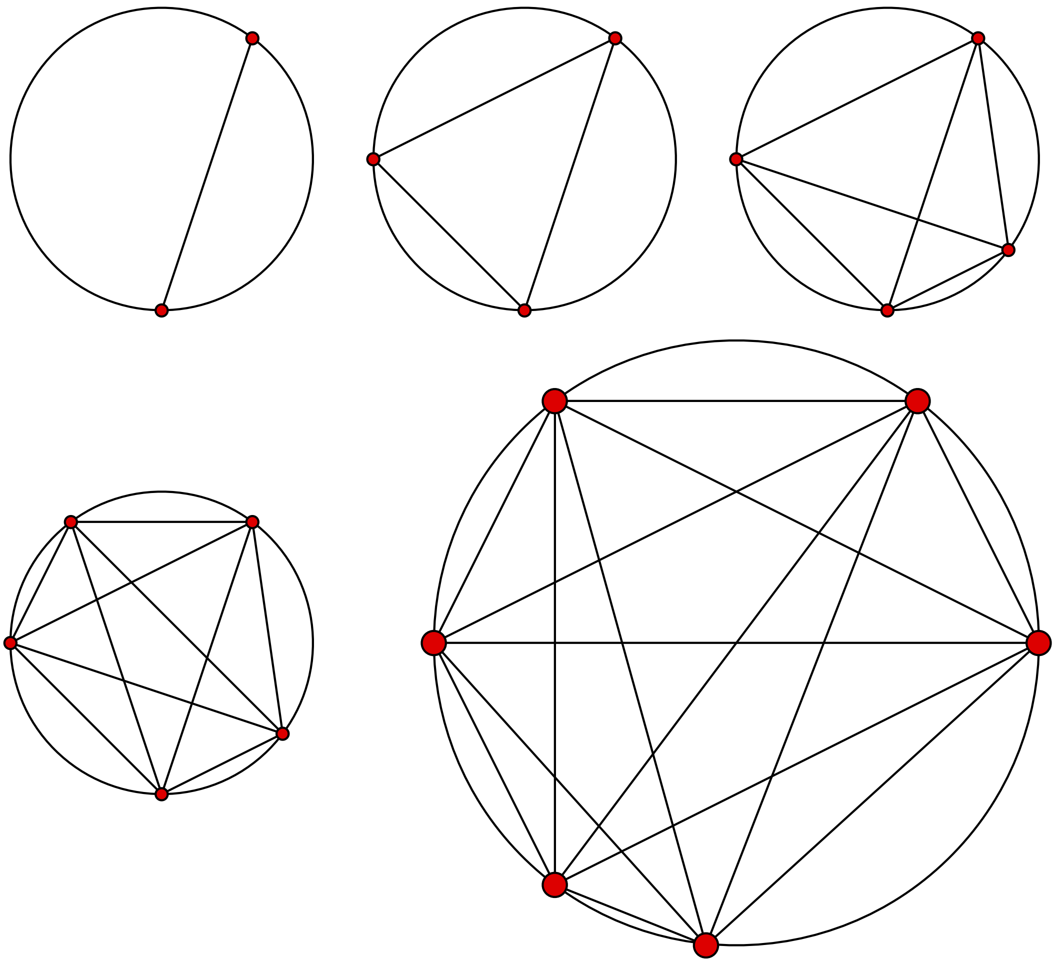
3.1 Always, Sometimes, Never

In mathematical investigations, a lookout is kept for patterns be they numerical, geometrical or algebraic. Once a pattern is suspected, mathematics requires that a proof be found; a reason why the pattern holds and will continue to hold.

3.2 Finding a Counterexample

A pattern that seems to be there may not, in fact, always hold. The easiest way to show that an alleged pattern is an illusion is to find a counterexample.

For example, suppose that a circle has n distinct points on its circumference, each joined to the others in all possible ways by straight lines. It is conjectured that the interior of the circle then has a maximum number of regions given by 2^{n-1}. Investigate this conjecture with the aid of the following diagrams.



n	2	3	4	5	6
2^{n-1}	2	4			
From Diagram					

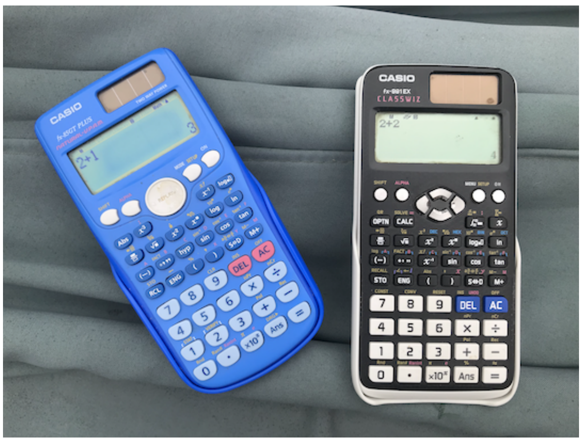
[6 marks]

3.3 When a Counterexample Can't Be Found

Either prove, or find a counterexample to the conjecture that, “if you add 1 to the product of four consecutive natural numbers the answer is always a perfect square”.

[6 marks]

~ Weapons of Mass Deduction ~



Photograph by Martin Hansen

3.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30 marks

Question 1

Give a counterexample to prove that the following statement is false;

For any pair of real numbers, x and y , if $x^2 = y^2$ then $x = y$

[2 marks]

Question 2

Give a counterexample to prove that the following statement is false;

For $a, b \in \mathbb{R}$, if $a^2 - b^2 > 0$ then $a - b > 0$

(Note that $a, b \in \mathbb{R}$ is saying that a and b are real numbers)

[2 marks]

Question 3

Give a counterexample to prove that the following statement is false;

For $x \in \mathbb{R}$, $0° \leqslant x \leqslant 360°$, $\cos x \leqslant 1 + \sin x$

[2 marks]

Question 4

For $x, y \in \mathbb{Q}$, $x \neq 0$, $y \neq 0$, give a counterexample to the following claim;

$$x^2 + y^2 \neq 1$$

(Note that the information $x, y \in \mathbb{Q}$ is saying that a and b are rational numbers)

[2 marks]

Question 5
AS-Level Examination Question from May 2018. Paper 1, Q2 (b), (Edexcel)
"If I add 3 to a number and square the sum, the result is greater than the square of the original number".
State if the above statement is always true, sometimes true or never true.
Give a reason for your answer.

[2 marks]

Question 6
Give a counterexample to prove that the following statement is false;
For any real numbers, x, if $|2x + 1| \leqslant 5$ then $|x| \leqslant 2$

[2 marks]

Question 7
Consider the following statement;

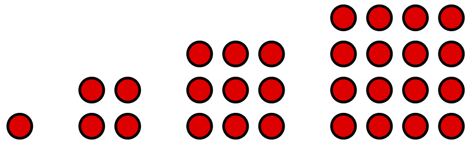
"For all $n \in \mathbb{N}$, $n^2 - n + 3$ is a prime number"

If true, prove it is true or, if false, find a counterexample.

[2 marks]

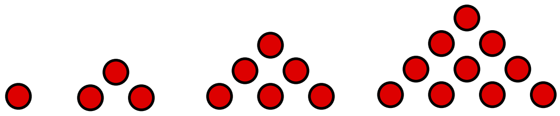
Question 8

The square numbers are so called because they are numbers that can be visualised as dots arranged into squares,



1 4 9 16

Similarly, the triangular numbers can be visualised as arrangements of dots into equilateral triangles,



1 3 6 10

The formulae for square numbers, S, and triangular numbers T are,

$$S_n = n^2 \qquad T_n = \frac{1}{2} n (n + 1) \quad \text{where } n \in \mathbb{N}$$

Consider the following statement;

"For all $n \in \mathbb{N}$, $8T_n + 1$ is always a square number"

If true, prove it is true or, if false, find a counterexample.

[5 marks]

Question 9

Consider the following statement;

"The difference between the squares of two consecutive
triangular numbers is always a cube"

If true, prove it is true or, if false, find a counterexample.

[5 marks]

Question 10

Examination Question from 2008, Q11 AH Maths (SQA)

For each of the following statements, decide whether it is true or false.
In each case, either prove it is true or find a counterexample.

(i) For $n \in \mathbb{N}$, if m^2 is divisible by 4 then m is divisible by 4.

[3 marks]

(ii) For $p, q \in \mathbb{N}$, the cube of any odd integer p plus the square of any even integer q is always odd.

[3 marks]

This document is a part of a **Mathematics Community Outreach Project** initiated by Shrewsbury School
It may be freely duplicated and distributed, unaltered, for non-profit educational use
In October 2020, Shrewsbury School was voted "**Independent School of the Year 2020**"
© 2025 Number Wonder
Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com